OPTIMIZING INTUITION

Uldarico Malaspina — Pontificia Universidad Católica del Perú

Vicenç Font — Universitat de Barcelona

In this paper we establish a link between the Onto-Semiotic Approach to mathematics cognition and instruction (Godino, Batanero and Font, 2007) and the Cognitive Science of Mathematics (Lakoff and Núñez, 2000). We use an a priori characterization of "optimizing intuition" as the context for reflection.

INTRODUCTION

One characteristic of the research community in mathematics education is its large diversity of different theoretical perspectives and research paradigms. Although diversity is not considered a problem but a rich resource for grasping complex realities, we need strategies for connecting theories or research results obtained with different theoretical approaches. Each theoretical perspective tends to privilege some reality dimensions over others. Thus, it is not always an easy task to find links among research questions, descriptions, methodologies and conclusions that are elaborated within different paradigms. A specific research effort is needed in this direction. In this paper we establish a link between the Onto-Semiotic Approach to mathematics cognition and instruction and the Cognitive Science of Mathematics. We use an a priori characterization of "optimizing intuition" as the context for reflection. The confirmation of the existence (or not) of this type of intuition should be the result of a posteriori research.

The available literature on intuition is reviewed, the theoretical framework is presented and the constructs of the theoretical framework are used to explain what is meant by optimizing intuition in this work.

LITERATURE REVIEW

The relationships between intuition and rigor have been studied and debated in the field of Mathematical Education. Fischbein (1994) defines the notion of intuition and analyzes the essential role that it plays in students' mathematical and scientific processes. He classifies intuitions in two ways: according to its functions and according to its origins, although he warns that these distinctions should not be considered as absolute ones. Using Fischbein's work as a reference, Tirosh and Stavy (1999) developed the theory of the intuitive rules which allows the analysis of students' inappropriate answers to a wide variety of mathematical tasks.

One possible classification of types of intuition is to consider the mathematical content to which intuition is applied. This classification of intuitions strengthens our question about the existence of an "optimizing intuition".

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 81-88. Thessaloniki, Greece: PME.

THEORETICAL FRAMEWORK

Below are summaries of the theoretical framework used in this work: *The onto-semiotic approach* and *the Cognitive science of mathematics*.

The onto-semiotic approach (OSA)

In Figure 1 we represent some of the different theoretical notions of the Onto-Semiotic Approach to mathematics cognition and instruction (Godino, Batanero and Font, 2007; Font and Contreras, 2008). Here mathematical activity plays a central role and is modelled in terms of systems of operative and discursive practices. From these practices the different types of related mathematical objects (language, arguments, concepts, propositions, procedures and problems) emerge building cognitive or epistemic configurations among them (see hexagon in Figure 1).

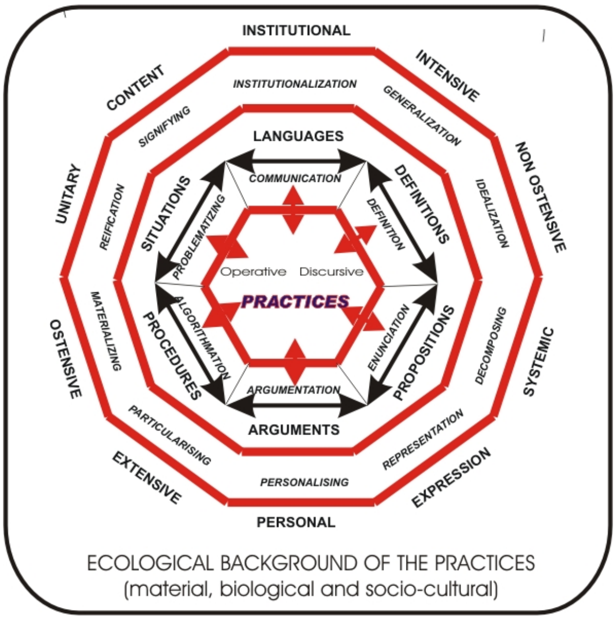

Figure 1. An onto-semiotic representation of mathematical knowledge

Problem-situations promote and contextualize the activity; languages (symbols, notations, graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices depend on the “language game” in which they participate, and might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown in Figure 1. Instead of giving a general definition of process the OSA opts to select a list of processes that are considered important in mathematical activity (those of Figure 1), without claiming that this list includes all the processes implicit in mathematical activity; this is because, among other reasons, some of the most important of them (for example, the solving of problems or modeling) are more than just processes and should be considered as hyper- or mega-processes.

Cognitive science of mathematics

Lakoff and Núñez (2000) state that the mathematical structures people build have to be looked for in daily cognitive processes such as image schemas and metaphorical thinking. Such processes allow us to explain how the construction of mathematical objects is supported by the way in which our body interacts with objects in everyday life. To reach abstract thinking we need to use more basic schemas that are derived from the very immediate experience of our bodies. These basic schemas, called *image schemas*, are used to give meaning, through *metaphorical mappings*, to our experiences in abstract domains. Lakoff and Núñez (2000) claim that metaphors create a conceptual relationship between the source domain and the target domain, and they distinguish two types of conceptual metaphors in relation to mathematics. A) grounding metaphors: which relate a source domain outside mathematics with a target domain inside mathematics; and B) linking metaphors, which have both their source and target domains in mathematics.

Lakoff and Núñez (2000) analyze four grounding metaphors whose target domain is arithmetic. These four metaphors offer an approximation to the relationship of order, which is vital to understanding the concepts of maximum and minimum. However, these authors also consider that the graphics of functions are structured through the metaphorical mapping of the "source-path-goal schema". Such a mapping conceptualizes the graphic of the function in terms of motion along a path, such as when a function is described as "going up", "reaching" a maximum, and "going down" again. In this way, the idea of the ups and downs of a road is essential to the understanding of the concepts of maximum and minimum.

The link between these frameworks

Several studies conducted within the OSA framework have developed theoretical connections between this approach and the theory of Lakoff and Núñez. Acevedo (2008) relates metaphorical processes to the sixteen processes shown in Figure 1, and does so by using the graphic representation of functions as the context for reflection. In the present work, we are interested in commenting in detail the understanding of what metaphorical processes are. That understanding results from observing these processes from the "unitary – systemic" duality proposed in the OSA (Acevedo, 2008).

In Lakoff's and Núñez's works, the *unitary – systemic* duality has a central role. On the one hand, the metaphor is unitary (A is B). On the other hand, the metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding of the target domain in terms of the source domain. Lakoff and Núñez develop the unitary–systemic duality for different metaphors. A good example can be the metaphor of the container, which according to Núñez (2000) is a metaphor used to structure the theory of classes. For this author, this metaphor is ontologic and unconscious and has its origin in every day life. (Núñez, 2000, p. 13):

Unitary: "Classes are containers"

Systemic:

Source Domain **Container Schemas**		Target Domain **Classes**
Interior of Container Schemas	→	Classes
Objects in Interiors	→	Class members
Being an Object of an Interior	→	The Membership Relation
An Interior of one Container Schema within a Larger One	→	A subclass in a Larger Class
The Overlap of the Interiors of Two Container Schemas	→	The Intersection of Two Classes
The Totality of the Interiors of Two Container Schemas	→	The Union of Two Classes
The Exterior of a Container Schemas	→	The Complement of a Class

Table 1. The metaphor "Classes are containers"

In fact, most research on metaphors has been mainly targeted at studying such a duality. In other words, given a metaphor, the source and the target domains are decomposed to determine what concepts, properties, relationships, etc. from the source domain are transferred to the target domain. The systemic vision of a metaphor leads us to understand it as a generator of new practices.

Because the OSA considers that on the one hand, among other aspects, an epistemic/cognitive configuration, depending on whether the adopted point of view is institutional or personal, has to be activated to do mathematical practices, and that on the other hand the systemic vision of the metaphor leads us to understand it as a generator of new practices, it is natural to ask ourselves the following question: *How is the metaphor related to the building components of epistemic/cognitive configurations?* The conclusion drawn by Acevedo (2008) on linking metaphors is that a linking metaphor projects an epistemic/cognitive configuration on another one.

The epistemic/cognitive configuration construct is used to explain and specify the structure that is projected onto the linking metaphors. Here there is a source domain that has the structure of an epistemic/cognitive configuration (depending on whether the adopted point of view is institutional or personal) and which projects itself onto a target domain that also has the structure of an epistemic/cognitive configuration. This way of understanding the preservation of the metaphoric projection improves upon the explanation of such preservation given by Lakoff and Núñez (2000), who only provide a two-column table in which — mainly— properties and concepts are mixed. The reader can intuit that the properties are projected on properties and the concepts on concepts.

The question that remains unresolved concerns what structure is projected in the case of a grounding metaphor. We believe that unlike the linking metaphors, only some parts of the epistemic/cognitive configuration are projected. The specific study of each grounding metaphor will allow these parts to be identified.

OPTIMIZING INTUITION IN THE PRESENT RESEARCH

Now we explain what we mean by optimizing intuition in our research. To achieve this goal let us considerer the following question as a context for reflection: Why are there people who consider it evident that a graphic which looks like a parabola that is shown to them (Figure 2(A)) has a maximum? To answer this question, we use three of the processes considered in Figure 1 (idealization, generalization, and argumentation), image schemas, and metaphorical mappings in the way they were applied in Acevedo's doctoral thesis (2008).

Above all, intuition has to do with the process of idealization (Font and Contreras 2008). Let us suppose that the teacher draws Figure 2(A) on the blackboard and that he talks about it as if he were displaying the graphic of a parabola, while simultaneously expecting that the students interpret the figure in a similarly way. The teacher and students talk about Figure 2(A) as if it were a parabola. However, if we look carefully at Figure 2(A), we can see that it is not actually a parabola. Clearly, the teacher hopes the students will go through the same idealization process with respect to Figure 2(A) and draw it on the sheet of paper as he has done. In other words, Figure 2(A) is an ideal figure (explicitly or implicitly) for the type of discourse the teacher and students produce about it. Figure 2(A), drawn on the sheet of paper, is concrete and ostensive (in the sense that it is drawn with ink and is observable by anyone who is in the classroom) and, as a result of the process of idealization, one has a non-ostensive object (the parabola) in the sense that one supposes it to be a mathematical object that cannot be presented directly. On the other hand, this non-ostensive object is particular. In the onto-semiotic approach, this type of "individualized" object is called an extensive object. Therefore, as a result of the process of idealization, we have moved from an ostensive object, which was extensive, to a non-ostensive object that continues to be an extensive.

The process of idealization is a process that duplicates entities, because in addition to the ostensive that is present in the world of human material experiences, it gives existence (at least in a virtual way) to an idealized non-ostensive object. Font, Godino, Planas and Acevedo (in press) argue that the key notion of object metaphor is central to the understanding how the teacher's discourse helps to develop the students' comprehension of non ostensive mathematical objects as objects that have "existence". In fact, classical authors, such as Plato, considered intuition precisely as a bridge between the space-temporal world of ostensive objects and the ideal world of non ostensive ones.

Intuition is also related to the generalization process because it can be considered as the process that allows us to see the general in the particular, a fact that is consistent with Fischbein's perspective (1994). In this case, there is also a generalization process according to which we consider this parabola to be a particular case of any curve that has "a similar shape to that of a parabola" (in the OSA this set is called an intensive object).

For classic authors such as Descartes the relationship between intuition and generalization is necessary to explain one of the basic characteristics of mathematical reasoning: the use of generic elements. In his fifth meditation Descartes proposes that it is necessary to consider a specific object for intuition, one which cannot refer to itself but to particular objects, in order to be able to act. Intuition allows us to grasp what is general in what is particular (capturing the essence). At all events it is not the goal of the present paper to explore in depth the problem of the relationship between the generic element and intuition; rather we simply wish to emphasize that any characterization of intuition should consider its relationship with generalization. Additionally, when intuition is related to the generalization derived from the use of generic elements, the complementary and dynamic relationship between rigor and intuition is highlighted. That is, intuition can be found in the intermediate steps of a proof or problem solution.

Given that intuition is usually considered as a clear and swift intellectual sensation of knowledge, of direct and immediate understanding, without using conscious and explicit logical reasoning, we can assume that in intuition there is no explicit argumentation even though there is an implicit inference. In the case shown in Figure 2 the inference could be, for instance, "as in the curve there is first a part that goes up and then a part that goes down, so there must be a point of maximum height".

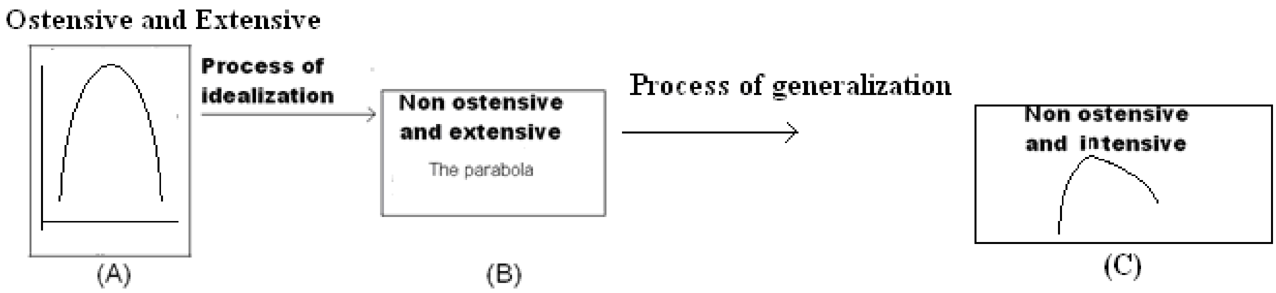

Figure 2. Idealization and generalization

The idea that intuition allows us to know the evident truth of specific mathematical propositions was a key element in what is known as the classical theory of truth, which remained valid until the appearance of non-Euclidian geometries. Briefly, that theory states that: 1) A statement is mathematically true if and only if the statement can be deduced from intuitive axioms; 2) Deduction from intuitive axioms is a necessary and sufficient condition of mathematical truth; 3) People perceive certain mathematical properties (e.g. axioms) as truths evident in themselves.

We thus propose the use of a vectorial metaphor in which the intuitive process is a vector with three components (any of which could be "null" in some cases).

Intuition = (idealization, generalization, argumentation)

With this metaphor it can be seen that intuition acts upon universal mathematical ideas (which are present through their associated ostensive objects) to get to results that are considered true without (or almost without) an explicit argumentation. In fact, the different ways in which intuition can be understood differ in the emphasis they place on each of the three components of the "intuition vector".

The task now is to find an explanation for the argumentation component in the specific case being dealt with here, i.e., how we can explain that it is evident that "because it first goes up and then comes down, there must be a maximum". We argue that there are reasons to assume that there is an *optimizing intuition* which leads this type of statement to be considered evident. Optimizing intuition basically has its origin in two types of everyday experiences. The first has to do with the fact that in everyday life we frequently have to face optimization problems, such as when we try to find the best way to go from one place to another (not necessarily the shortest), or when we try to make the best purchase, etc. This type of situations has an optimizing reasoning that seeks to find the best solution to a given situation.

The second type of experience is related to the fact that we are subjects who experience how certain physical characteristics, such as physical strength, health, etc., vary over time and pass through critical moments (maxima and minima). In this second type of experience we should also consider those related with the fact that we move frequently along roads which have ups and downs. We maintain that these two types of everyday situation allow us to make metaphorical mappings that contribute to the understanding of optimization problems. On the other hand, the very bodily experiences facilitate the appearance of the following optimizing image schema (Figure 3), which can subsequently be projected in more abstract domains. We maintain that the metaphorical mapping of these domains of experience (preferences, consumer, etc) and of the optimizing schema produces an intuitive understanding of optimization problems, and that it is this that allows us to answer intuitively the question with which we began this sub-section. The optimizing schema derived from the source-path-goal schema.

Using Fischbein's distinction between primary and secondary intuitions, as a reference, we believe that this kind of intuition would be of the primary type, one which remains as stable acquisitions throughout life and which due to the development of formal abilities, can gain in precision.

In terms of the epistemic/cognitive configuration one of the characteristics of the primary optimizing intuition is that the epistemic/cognitive configurations of the solution to optimization problems solved with optimizing intuition do not present the argument that justifies why the solution obtained is the optimum, given that it is considered evident that the found value is the optimum. To get more details you can refer to Malaspina (2008).

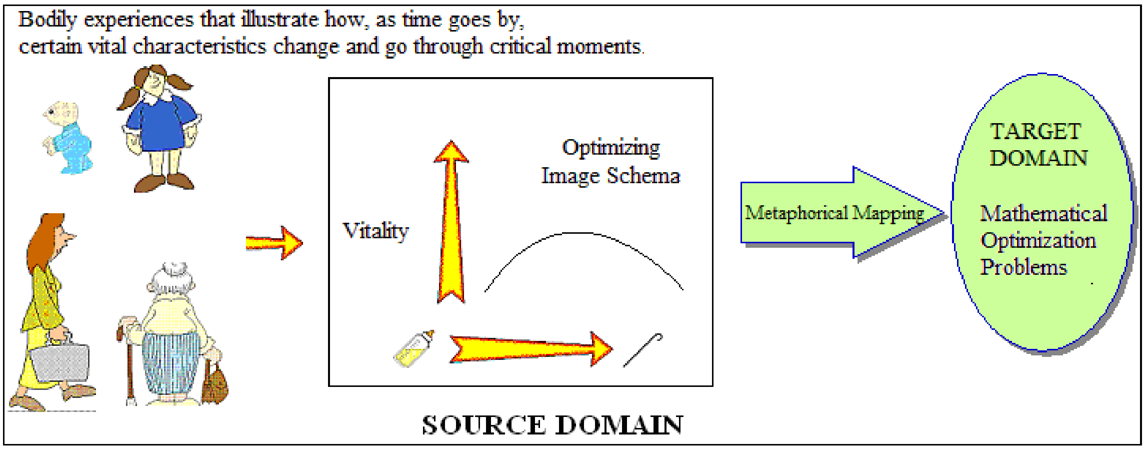

Figure 3. Metaphorical mapping of the optimizing schema

References

Acevedo, J. I. (2008). *Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones*. Unpublished Doctoral Dissertation. Barcelona, Spain: Universitat de Barcelona.

Fischbein, E. (1994). *Intuition in science and mathematics*, Reidel, Dordrecht.

Font, V. & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics*, *69*, 33-52.

Font, V., Godino, J.D., Planas, N. and Acevedo, J. I. (in press). The existence of mathematical objects in the classroom discourse, *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (CERME 6), University of Lyon, France.

Godino, J. D.; Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *Zentralblatt für Didaktik der Mathematik*, *39*, 127-135.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Malaspina, U. (2008). *Intuición y rigor en la resolución de problemas de optimización. Un análisis desde el Enfoque Ontosemiótico de la Cognición e Instrucción Matemática*. Unpublished Doctoral Thesis. Pontificia Universidad Católica del Perú.

Núñez, R. (2000). Mathematical idea analysis: GAT embodied cognitive science can say about the human nature of mathematics, in T. Nakaora and M. Koyama (Eds.). *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (vol.1, pp. 3-22). Hiroshima University, Hiroshima: Japan.

Tirosh, D. and Stavy, R. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational studies in Mathematics*, *38*, 51-66.