

A METHODOLOGY FOR THE DESIGN OF QUESTIONNAIRES TO EXPLORE THE MATHEMATICAL DIMENSION AND THE EPISTEMIC FACET OF DIDACTIC-MATHEMATICAL KNOWLEDGE OF TEACHERS

Luis R. Pino-Fan¹, Vicenç Font², Juan D. Godino³

¹University of Los Lagos, ²University of Barcelona, ³University of Granada

This paper presents some of the results obtained in a study that explored prospective secondary teachers' Didactic-Mathematical Knowledge (DMK) regarding the derivative. The specific focus was on mathematical dimension and key aspects of the epistemic facet of DMK. The study was carried out in three stages: 1) design of a questionnaire, which we called the DMK-Derivative Questionnaire; 2) pilot application of this questionnaire so as to develop its final version; and 3) application of the final version to a sample of Mexican prospective secondary teachers. Both the design of the questionnaire and the responses of the prospective teachers reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. The three mentioned stages are provided as a useful methodology for the design of instruments for assessing both the mathematical dimension and other dimensions of DMK.

Key words: teachers' knowledge, teacher training, onto-semiotic approach, derivative.

INTRODUCTION

As we noted in a previous study presented at the 36th Conference of the International Group for the Psychology of Mathematics Education (Pino-Fan, Godino, Font & Castro, 2012), the last three decades have seen a growing interest in determining the knowledge which teachers require to teach specific topics such as the derivative. Our understanding of this knowledge has been advanced through the work of authors such as Shulman (1986), Ball (2000), Ball, Lubienski and Mewborn (2001), and Hill, Ball and Schilling (2008). However, this research, along with that carried out by Fennema and Franke (1992), Llinares and Krainer (2006), Ponte and Chapman (2006), Philipp (2007), Sowder (2007), and Sullivan and Wood (2008), has led to a multifaceted view of the way in which knowledge for teaching is constructed. Indeed, there is no universal agreement as to the theoretical framework which should be used to describe mathematics teachers' knowledge. Furthermore, although progress has been made in terms of defining the categories of knowledge that mathematics teachers need in order to teach effectively, a number of questions remain to be resolved. For example, how, or according to what criteria, might we measure the knowledge corresponding to each of these categories? Having identified such criteria, are they then useful for developing and promoting among prospective teachers the kinds of mathematical

knowledge required by the different aspects of mathematics teaching? The research described in this paper seeks to address precisely these two questions.

In Pino-Fan et al. (2012) we presented the results obtained in the first part of a study that explored what we call the *epistemic facet* of prospective secondary teachers' didactic-mathematical knowledge about the derivative. The study as a whole involved three stages: 1) design of a questionnaire to evaluate teachers' knowledge, specifically as regards the epistemic facet of didactic-mathematical knowledge about the derivative; 2) analysis both of the results obtained in a pilot application of this questionnaire, as well as of the comments made by a number of experts who were consulted regarding the validity and reliability of the questionnaire; and 3) based on the information gathered in stage 2, development and application of a final version of the questionnaire, and analysis of the results obtained. This third stage also included interviews in order to obtain a more detailed understanding of prospective teachers' knowledge.

The present paper provides an overview of these three stages. However, as stage 1 and part of stage 2 have already been addressed in Pino-Fan et al. (2012) the emphasis here is on stage 2 and, especially, stage 3. The results obtained provide partial answers to the two questions that motivated the research.

THEORETICAL FRAMEWORK

The reference framework for this study is the Onto-Semiotic Approach (OSA) to mathematical knowledge and teaching, one that has been developed in several studies since 1998 (Godino & Batanero, 1998; Godino, Batanero & Font, 2007; Godino, Font, Wilhelmi & Lurduy, 2011; Font, Godino & Gallardo, 2012). More specifically, we make use of a model developed by Godino (2009), and refined in different works (Godino & Pino-Fan, 2013; Pino-Fan, Godino & Font, 2013a; Pino-Fan, Godino & Font, 2014), within the framework of the OSA, namely the *Didactic-Mathematical Knowledge (DMK)* model, which was designed to categorize and analyse the knowledge that teachers require to teach specific topics. The *DMK* model interprets and characterizes the teacher's knowledge from three dimensions (Pino-Fan, Godino & Font, 2014): *mathematical dimension*, *didactical dimension* and *meta didactic-mathematical dimension*. Each of these dimensions considers subcategories of knowledge, which, in turn, also include theoretical and methodological tools that allow operationalizing knowledge analysis regarding each subcategory. Furthermore, these dimensions, with their corresponding analysis tools, are involved in each of the phases proposed for the elaboration of Instructional Designs: preliminary study, design, implementation and evaluation.

In the case of the present paper the topic is the derivative, and the focus is on mathematical dimension and one of the six facets included in the didactical dimension of DMK model, the epistemic facet. DMK's mathematical dimension makes reference to the knowledge that allows the teacher to solve the problem or mathematical activity that is to be implemented in the classroom and link it with

mathematical objects that can later be found in the school mathematics curriculum. It includes two subcategories of knowledge: *common content knowledge* and *extended content knowledge*. The first subcategory, common content knowledge, is the knowledge of a specific mathematical object, which is considered as sufficient to solve problems and tasks proposed in the mathematics curriculum and in the textbooks of a certain educational level; it is a shared knowledge between the teacher and the students. The second subcategory, extended knowledge, refers to the knowledge that the teacher must have about mathematical notions that, taking the mathematical notions that are being studied at a certain time as a reference (for example, derivatives), come ahead in the curriculum of the educational level in question or in the next level (for example, integers in high school or the fundamental theorem of calculus in college). Extended content knowledge provides the teacher with the necessary mathematical foundations to suggest new mathematical challenges in the classroom, to link a certain mathematical object being studied with other mathematical notions and to guide students to the study of subsequent mathematical notions to the notion that is being studied (Pino-Fan, Godino & Font, 2014).

For its part, the *epistemic facet*, one of the facets involved in the didactical dimension of DMK, refers to specialized knowledge of the mathematical dimension. The teacher must have a certain amount of mathematical knowledge “shaped” for teaching; that is to say, the teacher must be able to mobilize several representations of a mathematical object, to solve a task through different procedures, to link mathematical objects with other mathematical objects taught at a certain educational level or from previous or upcoming levels, to comprehend and mobilize the diversity of partial meanings for a single mathematical object –that are part of the holistic meaning for such object (Pino-Fan, Godino & Font, 2011)–, to provide several justifications and argumentations, and to identify the knowledge at play during the process of solving a mathematical task (Pino-Fan & Godino, 2014).

For the mathematical dimension and the epistemic facet, two levels of analysis are proposed: 1) *Mathematical and didactic practices*, in other words, a description of the actions performed to solve the proposed mathematical tasks, so as to contextualize content and promote learning. 2) *Configuration of objects and processes*, that is, a description of the mathematical objects and processes involved in the mathematical practices under study, as well as those which emerge out of them. The purpose of this level is to reveal the complexity of those objects and meanings that are involved in teachers’ mathematical and didactic practices. This complexity is an explanatory factor in relation both to conflicts of meaning and the progression in learning.

METHOD

The research was based on a mixed methods approach (Johnson & Onwuegbuzie, 2004), since it was an exploratory study that examined both quantitative (level of accuracy of answers to questionnaire items: correct, partially correct and incorrect)

and qualitative variables (type of cognitive configuration activated when solving the tasks set). The three stages of the research were as follows.

Stage 1: design of the questionnaire

In this first stage we began by creating a bank of tasks involving the derivative. These tasks were drawn from various studies on the teaching of calculus. In order to select the tasks that would be included in the questionnaire we considered three criteria: 1) the questionnaire needed to cover the different meanings of the derivative, taking as a reference the holistic view of the derivative that is set out in Pino-Fan, Godino and Font (2011); 2) it had to capture movement between the different ways of representing both the function and its derivative; and 3) it had to reflect the type of didactic-mathematical knowledge that corresponds to the epistemic facet and mathematical dimension (common and extended). This first stage is described in detail in Pino-Fan et al. (2012).

Stage 2 (part one): pilot application of the questionnaire

Having designed the initial version of the questionnaire, which included seven tasks that reflected the three criteria, we then selected an intentional sample of prospective secondary teachers, all of whom were students on the degree in mathematics teaching offered by the Autonomous University of Yucatan (UADY) in Mexico. This degree course comprises eight semesters, and the 53 prospective teachers who made up the sample were enrolled in semesters six and eight. These students had already taken courses related not only to mathematical analysis (differential and integral calculus, differential equations, etc.) but also to the teaching of mathematical subject areas (geometry, differential and integral calculus, conceptual development of calculus, etc.). The questionnaire was administered at the beginning of February 2011, and took two hours to complete. The results of this pilot application are reported in Pino-Fan et al. (2012).

Stage 2 (part two): expert triangulation

In order to ensure the reliability and validity of the questionnaire it was subjected to a process of expert triangulation. By means of a survey, eight experts were asked to assess the relevance of the questionnaire items for evaluating each of the following aspects: 1) different meanings of the derivative; 2) representations activated both in the item statements and their possible solutions; and 3) the type of didactic-mathematical knowledge (corresponding to the epistemic facet and mathematical dimension) required to solve the tasks. In order to make any necessary improvements to the pilot version the experts were also asked for their opinion regarding any important content that was missing from the questionnaire, as well as the wording and comprehensibility of items. It should be noted that we use the word “expert” to refer us to researchers who have extensive trajectory in the research on the didactic of calculus and the teachers training. So, research professors from France, Mexico, Spain, Colombia and Portugal, participated in this study of the “expert triangulation”.

Stage 3: application of the final version of the questionnaire

The final version of the questionnaire, which we called the DMK-Derivative Questionnaire, was administered to 49 prospective teachers at the beginning of February 2012. These prospective teachers constituted the total number of students who, at that time, were enrolled in semesters six and eight of the degree in mathematics teaching offered by the UADY. As in the pilot the final version took two hours to complete, and the students to whom it was administered had, like their counterparts in stage 1 of the research (in 2011), already taken courses about mathematical analysis and how to teach calculus. One week after administration of the questionnaire, and once the results had been gathered, a series of semi-structured interviews were scheduled with the students. The aim of these interviews, which took into account the suggestions of the eight experts, was to explore in greater detail the knowledge and, specifically, the cognitive configurations used by the prospective teachers when solving the tasks set.

RESULTS AND DISCUSSION

In this section we discuss the results obtained in part two of stage 2 (expert triangulation) and in stage 3 (application of the final version of the questionnaire) of the study.

Stage 2 (part two): expert triangulation

The triangulation process involved eight experts, all of whom were internationally recognized for their research on calculus and how to teach it. In general, the DMK-Derivative Questionnaire was viewed positively by the experts, as illustrated by the following two comments:

“I think the questionnaire is very thorough. It’s worth noting that it measures not only the types of knowledge but also the way in which they have been acquired. I’m referring here to the difference between a highly mechanical or routine form of learning and one that is meaningful, where the derivative of a point has various representations” (Expert E7).

“...I would like to point out that the proposed tasks are rich and varied, and I think they are capable of measuring prospective secondary teachers’ didactic-mathematical knowledge about the derivative. I would also like to highlight the importance of the criteria defined in order to select the tasks: different meanings of the derivative; the use of different representations activated both in the item statements and the solutions to them; and the three components of prospective teachers’ didactic-mathematical knowledge (common, extended and specialized content knowledge)” (Expert E8).

The experts also made a number of observations regarding each of the tasks included in the questionnaire, notably the following:

“Some students may have a background in economics or business administration, where the derivative is associated with concepts such as marginality, and where it appears as the function of marginal utility... It would be useful to include a question that relates the derivative to economics...” (Expert E5).

“...it is necessary to include activities related to modelling, such as optimization problems or those involving the instantaneous rate of change. In my view it would also be important to include a problem in which verbal expressions play a more significant role ...” (Expert E1).

Along with other observations and suggestions these comments by experts E5 and E1 led us to include the tasks shown in Figure 1.

9. In a company the total cost of producing q units is given by the function $C(q) = \frac{1}{3}q^3 - 12q^2 + 150q + 2304$
- a) Find the functions that determine the mean total cost and the marginal cost.
 - b) Determine the marginal cost and the mean total cost when producing 3 and 6 units.
10. The kinetic energy of an object is directly proportional to the square of its velocity and it has been shown experimentally that the proportionality constant is half its mass. What is the rate of change in kinetic energy with respect to velocity when $v = 0 \frac{mts}{seg}$? Justify your answer.
11. Is it possible to find two numbers such that their sum is 120 and the product of one number and the square of the other number is a maximum? If so, what are these numbers? Justify your answer.

Figure 1. Three new tasks included in the final version of the DMK-Derivative Questionnaire

One of the experts (E8) proposed the inclusion of five new tasks so as to explore in greater depth the variety of representations. However, we did not follow this suggestion, as we agreed with a comment made by expert E7:

“Rather than add new tasks I think it would be better to complement the questionnaire with interviews that could provide more detailed information about the students’ responses. Of course, more tasks can always be added, but it is important that the questionnaire does not become too long.”

Stage 3: application of the final version of the questionnaire

In order to evaluate the quantitative variable ‘level of accuracy of the students’ answers’, each of the questionnaire items was scored as follows: 2 points if the answer was correct, 1 if it was partially correct and 0 if it was incorrect. Thus, the

maximum possible score for a student who offered a correct solution to all the items of all the tasks was 36. Figure 2 shows the distribution of scores obtained by the 49 prospective teachers when responding to the final version of the questionnaire, and also indicates that the mean score was 13.8 (38.3% of the possible total score). It should be noted that of the 49 students who responded to the questionnaire, only 28 (57.1%) scored higher than the mean, and none of them reached a score of 24. This means that 42.9% of these prospective teachers scored below the mean. The tasks which they found most difficult were the three shown in Figure 1, which had a difficulty index of almost zero.

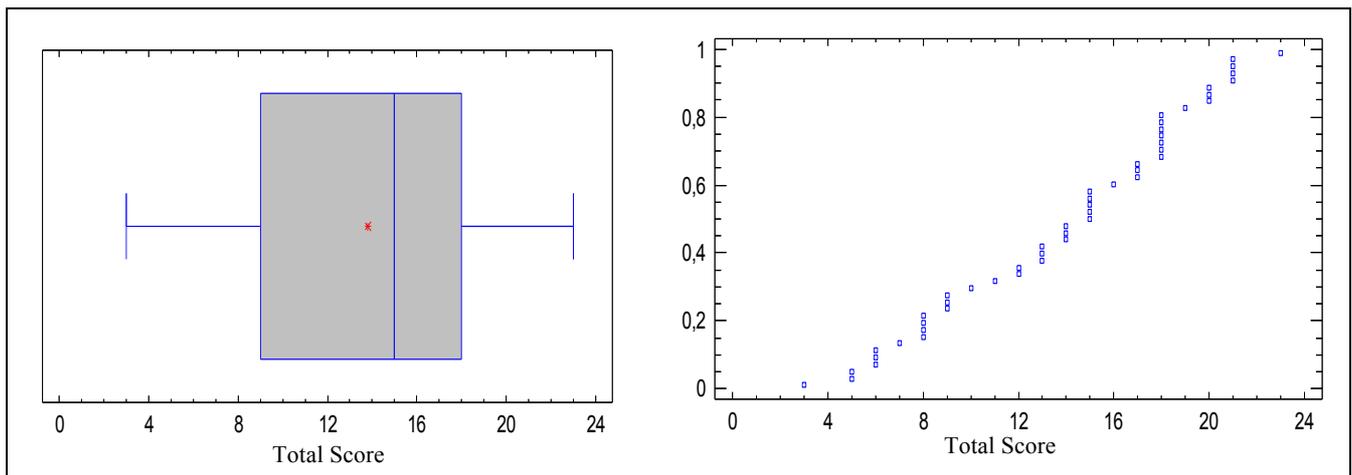


Figure 2: Distribution and mean of the scores obtained on the DMK-Derivative Questionnaire

The qualitative variable of interest in the present study was the *type of cognitive configuration*, which refers to the systematic description of primary mathematical objects (linguistic elements, concepts, propositions, procedures and arguments) and of the processes and interactions between them. The analysis of the cognitive configurations involved in the students' solutions, together with the data obtained through the subsequent interviews, revealed that 1) the prospective teachers did not make connections between the different partial meanings of the derivative; 2) when attempting to solve the tasks they found it difficult to make use not only of the elements considered to be part of specialized content knowledge (primary mathematical objects, processes and their links) but also of the epistemic facet in general; and 3) their grasp of the epistemic facet of didactic-mathematical knowledge would not be sufficient to enable them to manage adequately their future students' learning about the derivative. For reasons of space we are unable here to present one of the tasks so as to illustrate the type of qualitative analysis carried out. However, examples of the type of analysis, as well as of the cognitive configurations observed in the prospective teachers' answers, can be found in Pino-Fan, Godino and Font (2013 b) and Pino-Fan, Godino and Font (2014).

FINAL REFLECTIONS

This paper has described aspects of a research project carried out over the last three years and whose purpose was to explore the epistemic facet of prospective teachers' didactic-mathematical knowledge (DMK) about the derivative. Originally one of our aims was to examine and foster this kind of knowledge so that prospective teachers would be better equipped to teach the derivative. However, we then encountered the difficulty of how to explore all the different aspects of the mathematical knowledge that is required to teach the derivative. Consequently, we restricted our study to the mathematical dimension and the epistemic facet of DMK among prospective teachers.

The results of our research show that the variable and theoretical-methodological tool '*type of cognitive configuration*' activated in the prospective teachers' answers is useful for understanding the kind of didactic-mathematical knowledge they possess. Specifically helps us understand their mathematical knowledge (common, extended and specialized). This variable was analysed by means of a tool that we refer to as the '*configuration of primary mathematical objects and processes*', one which facilitates the analysis and categorization of certain features of the epistemic facet of prospective teachers' didactic-mathematical knowledge. Likewise, the dimensions and sub-dimensions proposed by the DMK, as well as the theoretical-methodological tools proposed to operationalize these dimensions, allow an approach to the answers of the questions raised at the beginning.

The design of the questionnaire used in this study, as well as the responses of prospective teachers to it, reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. Teachers need to become aware of this complexity during their training so that they will be able to develop and assess the mathematical competence of their future students.

Acknowledgements

This study was conducted within the framework of project EDU2012-32644 (University of Barcelona).

References

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51, 241-247.
- Ball, D. L., Lubienski, S. T. & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). Washington, DC: American Educational Research Association.
- Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A*

project of the National Council of Teachers of Mathematics (pp. 147-164). New York: Macmillan.

- Font, V., Godino, J. D. & Gallardo, J. (2012). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*. DOI: 10.1007/s10649-012-9411-0.
- Godino, J.D. (2009). Categorías de análisis de los conocimientos del profesor de matemáticas [Categories for analysing the knowledge of mathematics teachers]. *Unión, Revista Iberoamericana de Educación Matemática*, 20, 13-31.
- Godino, J. D. & Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in Mathematics Education. In: A. Sierpiska & J. Kilpatrick (Ed.), *Mathematics education as a research domain: A search for identity* (pp. 177-195). Dordrecht: Kluwer, A. P.
- Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, 39(1), 127-135.
- Godino, J. D., Font, V., Wilhelmi, M. & Lurduy, O. (2011). Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects. *Educational Studies in Mathematics*, 77(2), 247-265.
- Godino, J. D., & Pino-Fan, L. (2013). The mathematical knowledge for teaching. A view from onto-semiotic approach to mathematical knowledge and instruction. In B. Ubuz, Ç. Haser & M. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 3325 – 3326). Antalya, Turkey: CERME.
- Hill, H. C., Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Johnson, R.B. & Onwuegbuzie, A. (2004). Mixed methods research: a research paradigm whose time has come. *Educational Research*, 33(7), 14-26.
- Llinares, S. & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 429-459). Rotterdam: Sense Publishers.
- Philipp, R. (2007). Mathematics teachers' beliefs and affect. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Pub.
- Pino-Fan, L., Godino, J.D. & Font, V. (2011). Faceta epistémica del conocimiento didáctico-matemático sobre la derivada [The epistemic facet of mathematical and

- didactic knowledge about the derivative]. *Educação Matemática Pesquisa*, 13(1), 141-178.
- Pino-Fan, L., Godino, J. D. & Font, V. (2013a). Diseño y aplicación de un instrumento para explorar la faceta epistémica del conocimiento didáctico-matemático de futuros profesores sobre la derivada (1ª Parte) [Design and implementation of an instrument to explore the epistemic facet of didactic-mathematical knowledge of future teachers on the derivative (First Part)]. *REVEMAT*, 8(2), 1 – 49.
- Pino-Fan, L., Godino, J. D. & Font, V. (2013b). Diseño y aplicación de un instrumento para explorar la faceta epistémica del conocimiento didáctico-matemático de futuros profesores sobre la derivada (segunda parte) [Design and implementation of an instrument to explore the epistemic facet of didactic-mathematical knowledge of future teachers on the derivative (Second Part)]. *REVEMAT*, 8, Ed. Especial (dez.), 1 – 47.
- Pino-Fan, L., Godino, J.D. & Font, V. (2014). Una propuesta para el análisis de las prácticas matemáticas de futuros profesores sobre derivadas [A proposal for the analysis of prospective teachers' mathematical practices on derivatives]. *BOLEMA*, (In press). Available from: [http://webs.ono.com/vicencfont/index_archivos/CDM_derivada_11%20junio%202014%20\(2\).pdf](http://webs.ono.com/vicencfont/index_archivos/CDM_derivada_11%20junio%202014%20(2).pdf)
- Pino-Fan, L., Godino, J.D., Font, V. & Castro, W.F. (2012). Key epistemic features of mathematical knowledge for teaching the derivative. In Tso, T.Y. (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297-304). Taipei, Taiwan: PME.
- Ponte, J.P. & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 461-494). Rotterdam: Sense Publishers.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Charlotte (pp. 157-223), NC: Information Age Pub.
- Sullivan, P. & Wood, Terry (Eds.) (2008). *The international handbook of mathematics teacher education. Volume 1: Knowledge and beliefs in mathematics teaching and teaching development*. Rotterdam: Sense Publishers.