DIMENSIONAL ANALOGY AND COORDINATE SYSTEMS

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Dimensional analogy as a technique and different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in the sciences, engineering and art. As part of the process of the construction of an epistemic network for the subject, the identification of objects and their dualities that emerge from this mathematical activity was carried out. The context of reflection consists of the responses of multivariate calculus students to two questions at the end of a semester of a multivariate calculus course.

WHY DIMENSIONAL ANALOGY?

In the familiar 3-dimensional space that we live in, there are three pairs of cardinal directions: up/down (altitude), north/south (latitude), and east/west (longitude), usually labelled x, y, and z respectively. The possibility of spaces with dimensions higher than 3 was presented by Bernhard Riemann in his 1854 habilitation dissertation *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the Hypotheses that Underlie the Foundation of Geometry) at the University of Göttingen. This work was a landmark in the formal academic activity of geometry in higher dimensions, including, in particular, four dimensions. A space of four spatial dimensions has an additional pair of cardinal directions, usually labelled w.

To understand the nature of four-dimensional space, a device called *dimensional analogy* is commonly employed. Dimensional analogy is the study of how $(n - 1)$ dimensions relate to n dimensions, and then inferring how n dimensions would relate to $(n + 1)$ dimensions. Emphasis is made on the fact that dimensional analogy, as used in this study, is a mathematical concept, and should not be confused with analogy as a cognitive mechanism or as a linguistic construct.

The relationship between spatial dimension as a geometric concept, and the algebraic representation of dimension as lists of coordinates, has been recorded in textbooks as well as books and articles of general mathematical circulation, both pure and applied (Banchoff, 1996; Doran & Lasenby, 2007). The abstract relations are present from the beginning levels of algebra, but the present study privileges the multivariate calculus course where.

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> "[…] It is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace,… find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems". (Montiel, Wilhelmi, Vidakovic, Elstak, 2009c, 140–141)

On the Unidata site (http://www.unidata.ucar.edu/software/netcdf/coords/0050.html), it is clear that the need to reach common understandings, conventions and representations in applications ranging from earth system science, flow problems in physics and robotics to computer graphics is a 'hot topic'. Methods for solving three-dimensional flow problems with the aid of two dimensional solutions require the analytical and geometrical notions that are, in most educational systems, first introduced in a multivariate calculus course. For example, it is necessary to understand the concept of dimension and use dimensional analogy to understand Green's theorem, part of the standard content of the multivariate calculus course.

Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent but it is in this subject that, for the first time, students must learn operations that are dimension-specific (such as the cross product) and make generalizations in terms of dimensions and their algebraic and geometric representations, which require flexible mathematical thinking. This was the reason that motivated, in 2007 our initiation of a research project whose objective is to determine under what conditions this particular mathematical knowledge is constructed and communicated. (Montiel, Vidakovic & Kabael, 2008; Montiel, Wilhelmi, Vidakovic & Elstak, 2009a; Montiel, Wilhelmi, Vidakovic & Elstak, 2009b; Montiel, Wilhelmi, Vidakovic & Elstak, 2009c; Montiel, Wilhelmi, Vidakovic & Elstak, 2010; Font, Montiel, Vidakovic & Wilhelmi, 2010).

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

> "The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry" (Tall, 1995, 168).

In a previous work (Font, Montiel, Vidakovic & Wilhelmi, 2010) we have described Dimensional Analogy by means of three different theoretical perspectives**:** the theory of embodied mathematics, known as the *Cognitive Science of Mathematics* (Lakoff and Núñez, 2000), *APOS Theory* (Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas, 1996) and the *Onto-Semiotic Approach* (OSA) (Godino, Batanero and Font, 2007). In the present work our objective is to illustrate how Dimensional Analogy emerges in the mathematical activity of university students in a third semester (multivariate) calculus course. It is mainly the OSA that provides the framework and language for this study. Figure 1 (Font and Contreras, 2008, p. 35) is a diagram that synthesizes the mathematical objects and dualities associated with them, as used in this Theory.

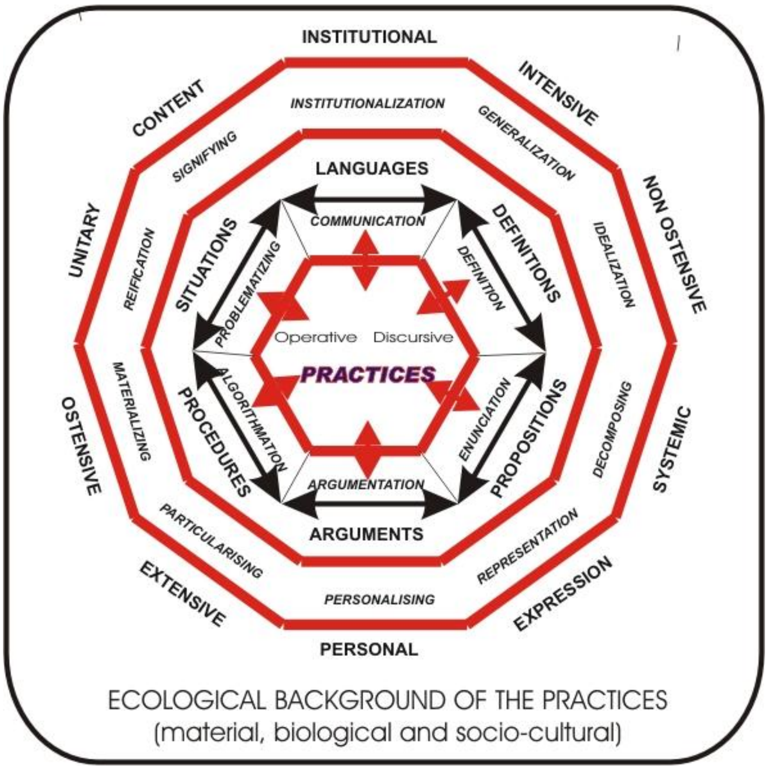

Figure 1: An onto-semiotic representation of mathematical knowledge

CONTEXT, METHOD AND INSTRUMENT

The context of the present study is multivariate calculus (calculus III) as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Seven students were interviewed once in two groups, the first consisting of four students and the second of three. The interviews were video-recorded. Each interview was approximately an hour long. The students were first given a questionnaire, which is included in this text, on which they wrote down their responses, and they were then asked to explain them. The interviewers used a common protocol to ask specific 'probing' questions. As final grades for the course had still not been submitted, another of the authors, who was the professor of the course, did not participate in the interviews, so that the students would not feel under any pressure in terms of their grades. The following table contains the questions presented to the students.

*Question 1***.** In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes. Draw them. Why are they planes and not just points or lines? In cylindrical coordinates, what are the three surfaces described by the

equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch. In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ Sketch.
*Question 2***.** What is the name of the following surface expressed as the polar function: $z = f(r, \theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \leq r \leq 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?

Table 1: Questionnaire

Preliminary analysis

The first question was included to detect the students' geometrical transition to 3D-space where, in the rectangular context, much emphasis was placed at the beginning of the course on the coordinate planes and the octants. These answers were important to begin to detect the process of dimensional analogy. The interview protocol included the question of why the equations represented planes, and not just points or lines. Although we have been unable to discover any literature on the subject, through informal discussions and comparisons it has been noted that the average student has difficulty with associating the algebraic equation, say, $y = a$, with a plane parallel to the *xz*–plane, or the actual *xz*–plane if $a = 0$. The protocol also indicated that, if the sketch was correct, to ask how the angle θ 'turns into' a plane.

The second question asked for the names of the quadric surface, but expressed algebraically as $f(r,\theta) = z$ instead of as the 'formula' that is taught when learning to identify quadric surfaces. Then the students were asked to find the volume of the solid by triple integration which connects volume to a triple – not a double – integral. The question was raised, as specified in the protocol, as to how the process of finding that volume could be carried out using a double integral and polar coordinates. This was done with the intention of detecting the use of dimensional analogy on the part of the students, in the context of the cylindrical and polar coordinate systems.

These first two questions play a methodological role in a long-term research project; that is, they constitute a mutual validation of the results obtained in this phase, and in two previous phases (Montiel et al. 2008, 2009c). The groups of students are homogeneous in the three investigations, and the instructional process is comparable (the same institution, the same intended meaning through the same instructor). For this reason, the discussion of the results is based not only in the onto-semiotic analysis (the study of the socio-epistemic configurations), but in the comparison and analysis of the data among these similar groups.

ACTUAL STUDENT ANSWERS TO THE QUESTIONS

Question 1

As a reply to the question of why the equations in rectangular coordinates represented planes and not lines, S3 said "For each plane there's only a restriction in one dimension, so that dimension is throughout the whole plane. Each other dimension can be anything, that's how you get just one infinite plane". The deictic signs that accompanied the verbal expression can be used to classify the relations that were established as a representational semiotic function, going from the equations, as the expression, to the sketches and gestures as content (in particular, the communication of the 'infinite plane' was clearly done by spatial gestures).

All students illustrated this situation by gesturing with their hands a subset of space (a plane) that seemed to be equidistant to their imagined points x_0, y_0 or z_0. There was only one student, S5, who initially had difficulties in verbalizing her explanation even though she had correct sketches of these planes on the paper. An interchange with the two interviewers (I1 and I2) and S5, when asked why the equations represented planes, will be presented.

Student 5: I don't think I know how to verbalize why they're planes and not points.

Interviewer1: But why do you see $x = x_0$ as parallel to the *yz*–plane?

Interviewer2: For the record, you're looking at the room.

S5: I'm looking at *z*, *y* and *x*.

I2: You're pointing, what exactly are you pointing at?

S5: The corners, I'm pointing at the corners to get my head around it in space.

On the other hand, S7 used dimensional analogy to explain:

S7: When we have 2 dimensional, say $x = 2$. We fix *x* at 2 and *y* could be anything. Now we have 3D, another variable which is *z* . Instead of say, $x = 3$ and *y* going on forever, it would be *z* going up and down forever as well.

In the context of the cylindrical coordinate system S3 offered his interpretation:

S3: $r = r_0$ would be a cylinder of infinite height, $\theta = \theta_0$, would be a slice of the cylinder and $z = z_0$ is an infinite plane, no, not a plane, it's an infinite disk at z_0.

I1: What is the difference between an infinite disk as opposed to a plane?

S3: It's just the coordinate system that you use, it's not rectangular coordinates any more, it's polar coordinates.

I2: But why think of it as a disk? I could think of it as an oval or a rectangular thing.

S3: It has no restriction on θ, it turns into a plane with an angle of 2π, the radius has no restriction, so it goes on and on.

S1: It has to do with how you build the plane in your head. If you take one *r* value at a time, it's just concentric disks. It's just the shape of the space.

As was established in the conceptual framework, the objects that have emerged, such as language, situations, procedures, definitions, reflect their duality. In particular, the duality ostensive (symbols, graphs, gestures) or non-ostensive, that is, conceptual or mental, can be detected in the chosen fragments related to question 1, through the semiotic functions with their expression and content.

Question 2

In this second question the students were asked to set up a triple integral to find volume. The students had been motivated geometrically to understand double integration as a means of calculating volume and the triple integral, in a spatial sense, as the geometric notion of hypervolume under some hypersurface in 4D-space.

The semiotic function that describes the mathematical activity is compound. First, the expression is a statement in mathematical English with symbolic language embedded in it, and the content is an integral set up which captures the meaning of the expression. Then this content (the triple integral) turns into the expression, whose meaning is a number that represents the volume asked for. This question stimulated dimensional analogy:

S7: To set up the triple integral, usually I understand it more as a double integral, where the function $f(r,\theta)$ would lie inside the double integral, and then I could visualize.

I1: If you were to use a double integral, how would you set it up?

S7: I would put *r* inside, it would be $\int_0^{2\pi}\int_0^2 r(r)\,dr\,d\theta$. It's the compensation from the polar coordinates. This is the function, and then we just worry about what's happening on the *xy*–plane. I understand it much more than to have it bounded above and below by another *z*–plane (sic). I can't imagine what it would look like with a triple integral.

I1: If you do the double and triple should you get the same answer?

S7: Yes, definitely.

The intensive/extensive and ostensive/non-ostensive dualities are apparent in this interchange, as the student makes his dimensional analogy. The extensive, with its ostensive representation in symbolic language, led to language objects that referred to a type (extensive), in this case the double integral as representing the volume under the surface. However, S4 also used dimensional analogy by 'going up' a dimension:

S4: When you have an actual function in double integrals that is the volume.

I1: Actual function?

S4: With triple integration, if we had a function it would be a hypervolume, but we don't have a function so it is just a volume.

However, S4 did not recognize the quadric surface and set up the integral with the radius of the cone as a constant. This reflects that she did not grasp the meaning (institutional), as an extensive, for setting up the triple integral to find volume, even though there is an 'actual' function ($z = \sqrt{x^2 + y^2}$).

CONCLUSIONS

The generic notion of representation is central in the cognitive and instructional processes involved in communicating Dimensional analogy. Dimensional analogy is used by the students, especially when dealing with the primary entities of language, situations and concepts, as can be witnessed in the analysis of question 2. However, this concept needs to be formalized and consciously incorporated as a technique in the communication of this mathematical subject, and others similar to it. The representational semiotic function was identified in several instances, where the expression consisted of the notion itself (of dimension), and the content was the actual analogy, manifested through gesticulation, graphical representations, double and triple integral signs, among other forms.

In the case of the different coordinate systems, definitions and procedures played a larger role. This is to be expected, as the change of coordinates was introduced as a procedure, while dimensional analogy plays more of a metaphoric role. Instrumental semiotic functions and structural (cooperative) semiotic functions were more noticeable for this same reason. The majority of students expressed that it was easier to use the calculus techniques and change the reference system, than to use purely geometric techniques, although their level of institutional success was inadequate.

This is a work in progress. Added to the development of the epistemic networks for topics in the multivariate context, there are plans to analyze the particular subject of change of coordinate systems as it is studied in the context of linear algebra, both introductory and advanced.

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