

# KEY EPISTEMIC FEATURES OF MATHEMATICAL KNOWLEDGE FOR TEACHING THE DERIVATIVE

Luis R. Pino-Fan<sup>1</sup>, Juan D. Godino<sup>1</sup>, Vicenç Font<sup>2</sup>, Walter F. Castro<sup>3</sup>

<sup>1</sup>Universidad de Granada, <sup>2</sup>Universitat de Barcelona, <sup>3</sup>Universidad de Antioquia

*In recent years there has been a growing interest in studying the knowledge that mathematics teachers require in order for their teaching to be effective. However, only a few studies have focused on the design and application of instruments that are capable of exploring different aspects of teachers' didactic-mathematical knowledge about specific topics. The present paper reports the results obtained following the application of a questionnaire designed specifically to explore certain key features of prospective, higher secondary-education teachers' knowledge of the derivative. The first part of the paper describes the design of this questionnaire.*

## BACKGROUND

The mathematical and didactic training of prospective teachers is an area of research that merits the attention not only of researchers in the field of mathematics education but also of educational authorities. Above all, this is because the development of pupils' mathematical thinking and competences is inherently dependent on their teachers' abilities.

One of the questions that have generated the most interest concerns how to determine the didactic-mathematical knowledge that is required to teach mathematics. In this context, the reflections and recommendations of Shulman (1986) and the studies by Ball (2000), Ball, Lubienski and Mewborn (2001) and Hill, Ball and Schilling (2008) have all helped to further our understanding of the different knowledge components that teachers need to acquire in order to teach effectively and foster their pupils' learning. However, a more detailed understanding of the knowledge required in order to teach mathematics needs to focus on specific topics, for example, the knowledge which a secondary teacher needs in order to teach the derivative (Badillo, Azcárate & Font, 2011). This paper reports some of the results obtained following the application of a questionnaire which, based on the model proposed by Godino (2009) for assessing and developing of the didactic-mathematical knowledge, was designed in order to explore key features of prospective secondary teachers' didactic-mathematical knowledge of the derivative.

## METHOD

The research is an exploratory study and uses a mixed methods approach; it involves the observation of both quantitative (level of accuracy of items: correct, partially correct, and incorrect answers) and qualitative variables (type of solution or cognitive configurations proposed by the prospective teachers). The latter, qualitative variable is

closely related to the type of didactic-mathematical knowledge which prospective, higher secondary-education teachers have about the derivative.

### **Subjects and context**

The questionnaire was administered to a sample of 53 students enrolled in the final modules (sixth and eighth semester) of the degree in mathematics teaching offered by the Universidad Autónoma de Yucatán (UADY) in Mexico. This is four-year degree (8 semesters). The Faculty of Mathematics of the UADY is responsible for training teachers to work at higher secondary or university level in the state of Yucatan (Mexico). The 53 students who responded to the questionnaire had studied differential calculus in the first semester of their degree course, and they had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). They had also studied subjects related to the teaching of mathematics.

### **The questionnaire**

The questionnaire, called the *Questionnaire regarding didactic-mathematical knowledge about the derivative (DMK-Derivative Questionnaire)*, comprises seven tasks and was designed in accordance with the model proposed by Godino (2009) for assessing and developing the didactic-mathematical knowledge. This model provides guidelines for categorizing and analysing teachers' didactic-mathematical knowledge in accordance to the onto-semiotic approach to knowledge and mathematics education (OSA) (Godino, Batanero & Font, 2007). The purpose of the questionnaire is to assess certain epistemic features of the didactic-mathematical knowledge (DMK) of prospective secondary teachers on the derivative. According to Ball and colleagues model (Ball, Lubienski & Mewborn, 2001; Hill, Ball & Schilling, 2008) this epistemic facet comprises three types of knowledge: *common content knowledge*, *specialized content knowledge* and *extended content knowledge*.

When designing the questionnaire, three criteria were considered in order to select the tasks that would be included in it. The first criterion was that the tasks should provide information about the extent to which a prospective teacher's personal understanding of the derivative was consistent with the global or holistic view of this mathematical object (Pino-Fan, Godino & Font, 2011). This was achieved by including items that activate different meanings of the derivative: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation. In this work we distinguish "instantaneous rate of change" that refers specifically to the "quotient" between two quantities of magnitudes, meanwhile "instantaneous rate of variation" refers to the "quotient" of real numbers with no reference to magnitudes. The "instantaneous rate of variation" is commonly known as the limit of the incremental quotient.

The second criterion was that the items selected had to reflect the different types of representations activated in the three sub-processes which, according to Font (2000), are involved in calculating the derivative function: 1) translations and conversions between the different ways of representing  $f(x)$ ; 2) the step from a representation of

$f(x)$  to a representation in the form  $f'(x)$ ; and 3) translations and conversions between the different ways of representing  $f'(x)$ . Consequently, the tasks included in the questionnaire bring into play the different types of representations that are involved in these three sub-processes, namely verbal description, graphical description, symbolic and tabulation (for both the function and its derivative).

The third criterion, which refers to the didactic-mathematical knowledge held by prospective teachers, considers the inclusion of three types of task: (1) those that require teachers to use their common content knowledge (solving a mathematical problem that would be set at the higher secondary level); (2) those that require specialized content knowledge (using different representations, different partial meanings of a mathematical object, solving the problem by means of various procedures, giving a range of valid arguments, identifying the knowledge that is brought into play when solving a mathematical problem, etc.); and (3) those that require extended content knowledge (generalizing tasks involving common or specialized knowledge, or making links to more advanced mathematical objects that appear in the curriculum). The next section presents an analysis of the aspects that are evaluated by the tasks included in the DMK-Derivative Questionnaire.

### **CONTENT ANALYSIS OF THE TASKS INCLUDED IN THE *DMK-DERIVATIVE QUESTIONNAIRE***

Due to space constraints this section only provides a detailed analysis of the knowledge assessed by five of the seven tasks included in the *DMK-Derivative Questionnaire*. However, both the ‘results and discussion’ and ‘final reflections’ sections include some discussion of all seven tasks in the questionnaire.

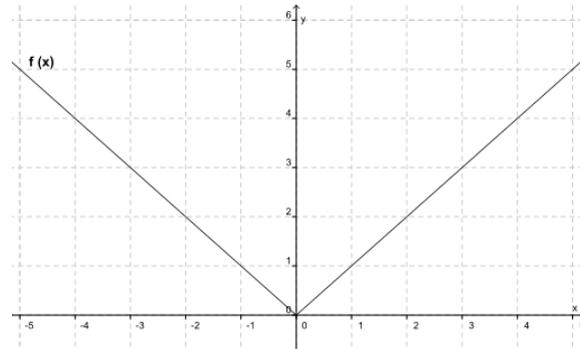
Task 1 is a classical question that has been used in a number of studies (Habre & Abboud, 2006; Bingolbali & Monaghan, 2008) to explore the meanings known by the students concerning the derivative. The question is: “What does the derivative mean to you?” As this is a ‘global’ question, prospective teachers are expected to provide a ‘list’ of possible meanings of the derivative. This task therefore explores the prospective teachers’ common knowledge regarding the meanings of the derivative.

Task 2 (Figure 1), which has been the object of several studies (Tsamir, Rasslan & Dreyfus, 2006), explores three types of knowledge that comprise the epistemic facet of didactic-mathematical knowledge about the derivative: 1) common content knowledge (item a), such that the prospective teacher should be able to solve the problem without needing to use various representations or arguments; 2) specialized content knowledge (items b and c), where in addition to solving the problem the teacher is required to use representations (graphs, symbols and verbal descriptions) and valid arguments that justify the procedures; and 3) extended content knowledge (item d), which entails generalization of the initial task about the derivability of the absolute value function at  $x=0$ , on the basis of valid justifications for the proposition “the graph of a derivable function cannot have peaks” by defining the derivative as the instantaneous rate of variation (limit of the increment quotient). The interpretations of the derivative as the

slope of the tangent line and the instantaneous rate of variation are associated with this task.

Task 2

Consider the function  $f(x) = |x|$  and its graph.



- For what values of  $x$  is  $f(x)$  derivable?
- If it is possible, calculate  $f'(2)$  and draw a graph of your solution. If it is not possible, explain why.
- If it is possible, calculate  $f'(0)$  and draw a graph of your solution. If it is not possible, explain why.
- Based on the definition of the derivative, justify why the graph of a derivable function cannot have 'peaks' (corners, angles)

Figure 1: Task 2 from the *DMK-Derivative Questionnaire*

Task 4, which is taken from Viholainen (2008), explores the specialized content knowledge of prospective teachers, as it requires the use of various representations (graph, verbal description, symbolic) and a range of justifications for the proposition “the derivative of a constant function is always equal to zero”, in which different interpretations of the derivative may be employed: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation.

Task 5 (Figure 2) appears to be the sort of exercise usually found in differential calculus books that are used at the higher secondary level, its solution being obtained by applying certain theorems or propositions about the derivative. Therefore, both item a) and item b) evaluate aspects of common content knowledge related to the derivative, where the latter is understood as the slope of the tangent line or the instantaneous rate of change, respectively. However, the main objective of Task 5 is to explore the associations that prospective teachers make between the different meanings of the derivative, and as such the task evaluates aspects of specialized content knowledge.

Task 5

Given the function  $y = x^3 - \frac{x^2}{2} - 2x + 3$

- Find the points on the graph of the function for which the tangent is horizontal.
- At what points is the instantaneous rate of change of  $y$  with respect to  $x$  equal to zero?

Figure 2: Task 5 from the *DMK-Derivative Questionnaire*

Finally, Task 7 (Figure 3), which has been adapted from the paper by Çetin (2009), provides information about the teachers' extended content knowledge, since it involves an approximation to the derivative of a function, described by values in the table, at point  $t=0.4$  by means of numerical values of the function. Task 7 is not the typical sort of problem that would be encountered at the higher secondary level, and it requires an understanding of the derivative as the instantaneous rate of change, and specifically as instantaneous velocity. This problem can be solved by various methods, for example, Lagrange's interpolating polynomial; this supports the categorization of this task as evaluator of the expanded content knowledge.

<u>Task 7</u>									
A ball is thrown into the air from a bridge 11 meters high. $f(t)$ denotes the distance that the ball is from the ground at time $t$ . Some values of $f(t)$ are shown in the table below:									
$t$ (sec.)	0	0.1	0.2	0.3	0.4	0.5	0.6		
$f(t)$ (m.)	11	12.4	13.8	15.1	16.3	17.4	18.4		
Based on the table, at what speed will the ball be travelling when it reaches a height at $t = 0.4$ seconds? Justify your chosen answer.									
a)	11.5 m/s	b)	1.23 m/s	c)	14.91 m/s	d)	16.3 m/s	e)	Another

Figure 3: Task 7 from the *DMK-Derivative Questionnaire*

## RESULTS AND DISCUSSION

In analysing the data obtained through administration of the questionnaire, we considered two variables: *the type of cognitive configuration* (i.e. type of solution proposed by the prospective teachers) and *level of task's accuracy* (i.e. correct, partially correct or incorrect). The analytic technique used with the first variable (type of cognitive configuration) was *semiotic analysis* (Godino, 2002), which provides a systematic description of both the mathematical activity carried out by the prospective teachers in solving the problems, and the mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) that were involved in their practice (Godino, Batanero & Font, 2007). The type of *didactic-mathematical knowledge* is closely related to the variable *type of cognitive configuration* associated to students' answers because the epistemic facet of didactic-mathematical knowledge depends on the presence or absence of the mathematical objects, their meanings and relations among them. These cognitive configurations have a didactic-mathematical nature due to the displayed tasks which have the same nature and therefore the prospective teachers should handle the didactic and mathematical knowledge.

Concerning the variable "level of accuracy" punctuations 2, 1 or 0 were assigned if the answers were correct, partially correct or incorrect correspondingly. Thus, the maximum possible score was 26. Twenty-four of the prospective teachers (45.3%) obtained a score higher than 13, but of these 24 only nine (17%) responded correctly to more than 67% of the questionnaire. The above information reveals that more than

50% of the students showed difficulties for solving questionnaire tasks. The mean score (12.4) obtained by the 53 prospective teachers and the distribution of their scores are shown in Figure 4.

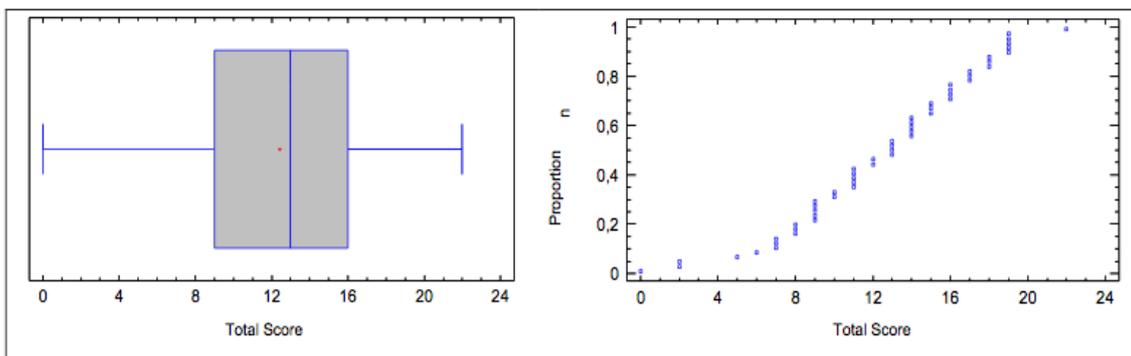


Figure 4: Boxplot and distribution of scores obtained on the *DMK-Derivative Questionnaire*

In general, the *DMK-Derivative Questionnaire* had an intermediate level of difficulty for the prospective teachers (Figure 5). The items they found most difficult were 2-d (Figure 1) and Task 7 as a whole (Figure 3). Task 1 and items 2-a, 3-a and 4-a were the easiest for them to solve.

Item	Difficulty Index	%
1 I-1	0.8679	86.79
2 I-2a	0.7547	75.47
3 I-2b	0.6038	60.38
4 I-2c	0.6415	64.15
5 I-2d	0.1321	13.21
6 I-3a	0.8491	84.91
7 I-3b	0.5660	56.60
8 I-4a	0.7547	75.47
9 I-4b	0.5849	58.49
10 I-5a	0.5660	56.60
11 I-5b	0.4528	45.28
12 I-6a	0.4528	45.28
13 I-7	0.1132	11.32
Mean:		0.56

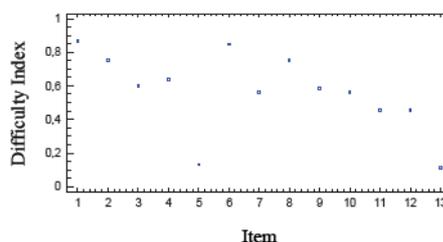


Figure 5: Difficulty index for the items on the *DMK-Derivative Questionnaire*

## FINAL REFLECTIONS

The analysis of the responses given by prospective teachers to the tasks included in the *DMK-Derivative Questionnaire* indicates that they had certain difficulties in solving the tasks related to common, specialized and extended knowledge about the derivative. The results obtained with respect to Task 4 show that these teachers performed better when solving tasks in which the derivative is understood as the slope of the tangent line. The results regarding Task 3 highlight the need to improve the advanced knowledge of prospective teachers, as this would help them to solve tasks such as this one. Due to the relationship between Task 4 and the type of knowledge it evaluates, it is clear that the prospective teachers lack certain aspects not only of specialized knowledge (use of different representations, use of different meanings of the derivative, solving the

problem through various procedures, giving a range of valid arguments to justify these procedures, etc.) but also of the common knowledge required to solve the task. Indeed, 56.6% of the teachers had problems (on Task 4) demonstrating, by means of the formal definition of the derivative, the proposition “the derivative of a constant function is always zero”. This suggests that they had yet to master the practice of demonstration when this involves using the derivative as the limit of mean rates of change. The results obtained in relation to Tasks 6 and 7 illustrate the difficulties which the prospective teachers experienced when they had to use the derivative as the instantaneous rate of change in a relatively complex situation. Here the *DMK-Derivative Questionnaire* revealed how common content knowledge is in itself not enough to deal with the kind of tasks that will emerge in the teaching context, for which teachers will also need a certain degree of both specialized and extended content knowledge. Indeed, the results show that not only did the prospective teachers lack certain aspects of specialized and extended knowledge, but also that there was a disconnection between the different meanings of the derivative (Tasks 1 and 5).

Both the design of the questionnaire and the responses of these prospective teachers reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. Teachers need to become aware of this complexity during their training so that they will be able to develop and assess the mathematical competence of their future pupils. In this regard, the aspects of specialized and extended knowledge that were lacking in these prospective teachers could hinder their ability to manage appropriately their future pupils’ mathematical knowledge about the derivative. The latter is supported by research showing that the mathematical knowledge of teachers has an effect on the achievements of their pupils (Ball, 1990; Wilson, Shulman & Richert, 1987). The lack of certain knowledge that was revealed in the present study highlights the need for specific training strategies to help prospective teachers develop the epistemic facet of their mathematical and didactic knowledge. The development of these training strategies should take into account the complexity of the global meaning of the derivative (Pino-Fan, Godino & Font, 2011).

### Acknowledgements

This study formed part of two research projects on teaching training: EDU2010-14947 (University of Granada) and EDU2009-08120 (University of Barcelona).

### References

- Badillo, E., Azcárate, C. & Font, V. (2011). Análisis de los niveles de comprensión de los objetos  $f'(a)$  y  $f'(x)$  en profesores de matemáticas [Analysing the extent to which mathematics teachers understand the objects  $f'(a)$  and  $f'(x)$ ]. *Enseñanza de las Ciencias*, 29(2), 191-206.
- Ball, D.L (1990). The mathematical understanding that prospective teachers bring to teacher education. *Elementary School Journal*, 90(4), 449-466

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51, 241-247.
- Ball, D. L., Lubienski, S. T. & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). Washington, DC: American Educational Research Association.
- Bingolbali, E. & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, 68 (1), 19-35.
- Çetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *PRIMUS*, 19(3), 232-244.
- Font, V. (2000). *Procediments per obtenir expressions simbòliques a partir de gràfiques. Aplicacions a la derivada* [Procedures for obtaining symbolic expressions from graphs: Applications in relation to the derivative]. Unpublished doctoral thesis: University of Barcelona.
- Godino, J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática [An ontological and semiotic approach to cognition in mathematics]. *Recherches en Didactique des Mathématiques*, 22(2/3), 237-284.
- Godino, J.D. (2009). Categorías de análisis de los conocimientos del profesor de matemáticas [Categories for analysing the knowledge of mathematics teachers]. *Unión, Revista Iberoamericana de Educación Matemática*, 20, 13-31.
- Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, 39(1), 127-135.
- Habre, S. & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior*, 25, 52-72.
- Hill, H. C., Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Pino-Fan, L., Godino, J.D. & Font, V. (2011). Faceta epistémica del conocimiento didáctico-matemático sobre la derivada [The epistemic facet of mathematical and didactic knowledge about the derivative]. *Educação Matemática Pesquisa*, 13(1), 141-178.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Tsamir, P., Rasslan, S. & Dreyfus, T. (2006). Prospective teachers' reactions to Right-or-Wrong tasks: The case of derivatives of absolute value functions. *Journal of Mathematical Behavior*, 25, 240-251.
- Viholainen, A. (2008). Finnish mathematics teacher student's informal and formal arguing skills in the case of derivative. *Nordic Studies in Mathematics Education*, 13(2), 71-92.
- Wilson, S.M., Shulman, L.S. & Richter, A (1987). 150 ways of knowing: Representations of knowledge in teaching. In L. Calderhead (Ed.), *Exploring teachers' thinking* (pp.104-1249). Sussex, England: Holt, Rinehart & Wilson.