Handbook of Mathematics Teaching Research:

Teaching Experiment - A Tool for Teacher-Researchers

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ONTO-SEMIOTIC TOOLS FOR THE ANALYSIS OF OUR OWN PRACTICE

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ABSTRACT
The analysis of educational practice is an important part of professional development, as it permits reflection on the various issues that condition the teaching and learning processes. The complexity of these processes requires the integration and application of theoretical tools that facilitate carrying out a systemic analysis. The perspectives required to analyze mathematical practice in the classroom are diverse, but can be usefully grouped in three categories: 1) the epistemic perspective, the goal of which is to provide adequate tools – what we shall refer to as "onto-semiotic" tools – for analyzing the objects and mathematical processes (personal and institutional) that intervene in an instructional process; 2) the normative perspective, which should furnish us with the tools for analyzing the socio-mathematical and social norms that regulate the study process; and, 3) the axiological perspective, which provides us with the tools to evaluate the efficacy of teachers' own practice. In this paper, we focus our study on the epistemic perspective and use the notions drawn from the onto-semiotic approach to mathematical cognition in analyzing a textbook task.

1. THE ONTO-SEMIOTIC APPROACH

The onto-semiotic approach to mathematical cognition tackles the problem of meaning and the representation of knowledge by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and socio-cultural theoretical frameworks (Ernest, 1998; Presmeg, 1998; Sfard, 2000; Radford, 2006). It assumes certain socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is implied and is dependent on the cultural institution and the social context of which it forms a part.

In Figure 1, we represent some of the different theoretical notions of the onto-semiotic approach for mathematical knowledge (Godino, Batanero & Roa, 2005; Font & Ramos, 2005; Font & Godino, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhelmi, 2006; Godino, Batanero & Font, 2007; Font, Godino & D'Amore, 2007). Here mathematical activity plays a central role and is modeled in term of systems of operative and discursive practices. From these practices emerge different types of mathematical objects that are related, building cognitive or epistemic configurations between them (the hexagon in Figure 1), such as: (i) language (terms, expressions, notations, and graphs); (ii) situations (problems, extra or intra-mathematical applications, exercises, etc.); (iii)concepts, given by their definitions or descriptions (number, point, straight line, mean, function, etc.); (iv) propositions, properties or attributes; (v) procedures (operations, algorithms, techniques); and (vi) arguments used to validate and explain the propositions and procedures (deductive, inductive, etc.). Finally, the objects that appear in mathematical practice and those that emerge from these practices depend on the language game in which they participate (Wittgenstein, 1953), and might be considered in terms of the five facets of dual dimensions (the decagon in Figure 1):

Personal-institutional. Institutional objects emerge from systems of shared practices within an institution, while personal objects emerge from the specific practices of an individual. "Personal cognition" is the result of individual thinking and activity when solving a given class of problem, while "institutional cognition" is the result of dialogue, agreement and regulation within the group of subjects belonging to a community of practices.

Ostensive-non ostensive. Mathematical objects (both at personal and institutional levels) are, in general, non perceptible. However, they are used in public practices through their associated *ostensives* (notations, symbols, graphs, etc.). The distinction between ostensive and non ostensive is related to the language game in which they take part. Ostensive objects can also be imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Extensive-intensive (example-type). An *extensive* object is used as a particular case (a specific example, i.e. the function $y = 2x+1$) of a more general class (i.e. the family of functions $y=mx+n$), which is an *intensive* object. The extensive/intensive duality is used to explain a basic feature of mathematical activity: the use of generic elements (Font & Contreras, 2008). This duality allows us to focus our attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge.

Unitary-systemic. In some circumstances mathematical objects are used as unitary entities (assumed to be previously known), while in other circumstances they are seen as systems that can be decomposed in order to be studied. For example, in teaching addition and subtraction the decimal number system (tens, hundreds, etc.) is considered as known, or as unitary entities. However, in the first grade, these same objects have to be dealt with as systemic and complex objects to be learned.

Expression-content: These are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are not isolated, but they are related in mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These dualities, as well as the objects, can be analyzed from a process-product perspective, which leads us to the processes in Figure 1 (Font & Contreras, 2008). In an onto-semiotic approach, the intention is not to provide a definition of a "process" from the outset, as there are many different types of processes: we can speak of a process as a sequence of practices, in terms of cognitive processes, metacognitive processes, processes of instruction, processes of change, social processes, etc. These are very different processes and, perhaps, the only characteristic that many of them may have in common is the consideration of the "time" factor and, to a lesser extent, the "sequence" in which each member takes part in the determination of the following step. For this reason, in the onto-semiotic approach, instead of offering a general definition of the process, we opt for the selection of a list of processes that can be considered important in mathematical activity (those included in Figure 1), without making any claims to include all the processes implicit in mathematical activity, nor for that matter even the most important, since, among other reasons, some of the most important (for example, the process of understanding, the solving of problems or modeling) rather than being processes, should be considered hyper- or mega-processes:

Figure 1. An onto-semiotic representation of mathematical knowledge.

In earlier works using the onto-semiotic approach to mathematical knowledge (D'Amore, Font & Godino, 2007; Font & Contreras, 2008; Font & Godino, 2006; Font, Godino & Contreras, 2008; Godino & Batanero, 1994; Godino, Bencomo, Font & Wilhelmi, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhelmi, 2006; Godino, Font, Wilhelmi & Castro, 2008), the authors proposed five levels for analyzing the study processes: (1) analysis of the types of problems and systems of practices; (2) analysis of configurations of mathematical objects and processes; (3) analysis of the didactic trajectories and interactions; (4) analysis of the system of norms and metanorms; and (5) evaluation of the didactical suitability of the study process.

Applying level 1 to a study process leads to describing the sequence of mathematical practices. Carrying out a practice mobilizes distinct elements, that is, an agent (institution or person) that develops a practice and a medium where it is developed (there may be other agents, objects, etc. in this medium). Due to the fact that an agent carries out practices oriented towards solving situation-problems, it is necessary to consider, among other aspects, mathematical objects and processes that make these practices possible; which is done in level 2. The final part of this second level of analysis is to describe the complexity of the mathematical practices, taking into consideration the diversity of objects and processes, as well as their typologies.

Given that the study of mathematics usually takes place under the direction of a teacher and includes interaction with other students, the didactic analysis should progress from the situation-problem and the mathematical practices necessary for solving it (level 1) to the configurations of mathematical objects and processes that make these practices possible (level 2), and from there to studying the interactions between teachers and students. In our case, and given the large diversity of didactic interactions that occur in any study process, for level 3 we focus on the interactions regarding easily identifiable semiotic conflicts (in the sense that their identification admits an easy triangulation of perspectives, at least those given by the three authors). In level 4, we will consider which mathematical practices and interactions are conditioned and supported by a set of norms and metanorms that regulate the actions and that are to be analyzed.

The four levels of analysis described above are tools for descriptive and explicative didactics, as they serve to understand and respond "What happened here, and why?" However, they do not evaluate the pertinence of the mathematical instruction process, nor determine guidelines for improving the design and implementation of this process. Mathematical didactics should not be limited to mere description, but should aspire to improving the orchestration and development of study processes. Thus, there is a need for criteria of suitability or appropriateness that permit evaluation of the instruction processes carried out and "guide" their improvement. In this model, level 5 attempts to provide such an evaluative perspective.

These levels are the result of a theoretical synthesis of various partial analyses that have been developed in the research area of mathematics education. For example, level 4 is proposed for integrating aspects of the analysis of socio-mathematical norms developed by socio-cultural perspectives in mathematics education (Civil & Planas, 2004; Cobb & McClain, 2006; Font & Planas, 2008; Stephan, Cobb & Gravemeijer, 2003; Yackel & Cobb, 1996). The levels of analysis proposed by the onto-semiotic approach have been designed for the development of a complete didactic analysis that permits describing, explaining and evaluating study processes. However, further analysis of some of the levels is strongly affected by the type of episode, which in some cases may even become an obstacle. As for level 5, in order to evaluate the didactical suitability of a study process (in accordance with the idea of didactical suitability developed by Godino, Bencomo, Font & Wilhelmi, 2006), a broad longitudinal analysis is needed, which the analyses of levels 1, 2, 3 and 4 applied to a brief classroom episode do not provide. This does not exclude the possibility of carrying out a partial evaluation of the suitability of a study process, keeping in mind, for example, the suitability of the interaction observed in the application of level 3. As for level 4, since the norms are inferred from regularities observed in the study process, their identification in a brief episode may be seen as questionable; despite this, a plausible inference of norms and metanorms can be developed, keeping in mind data obtained when levels 1, 2 and 3 were applied and assuming that these data are local.

In this paper we propose to apply only levels 1 and 2 using one task taken from a textbook.

2. VORONOI DIAGRAMS AS A CONTEXT FOR REFLECTION

As a context for reflection, we shall use the following task taken from *Geometry with Applications and Proofs* (Goddijn, Kindt & Reuter, 2004, part. I, 5) published by the Freudenthal Institute.

Task 1. In the desert
Below you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at *J*. You are very thirsty and you only brought this map with you.

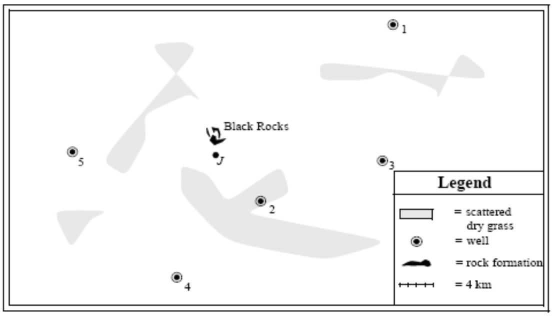


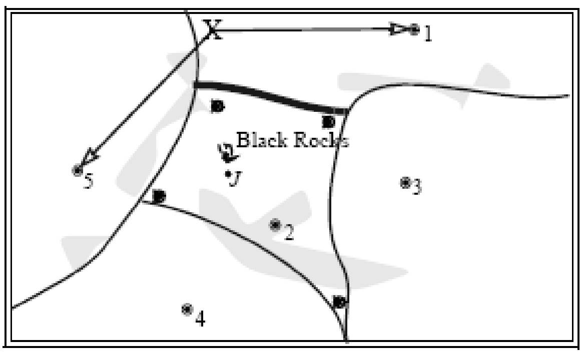
Figure 2. In the desert

1 a. To which well would you go for water?
That choice was not difficult. Of course, you would go to the nearest well.
b. Point out two other places from where you would also go to well 2. Choose them far apart from each other.
c. Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest.
d. What can you do when you are standing exactly on the edge of two different domains?
e. Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells.
f. In reality the desert is much larger than is shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?
g. The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?
h. What kind of lines are the edges? Straight? Curved?

In this exercise you just partitioned an area according to the *nearest-neighbor-principle*. Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then. Next we will investigate the simple case of two wells, or actually two points, since we might not be dealing with wells in other applications.

Solution proposed by the authors of the book (Goddijn, Kindt & Reuter, 2004, I-91):

1: In the desert
1 a. 2
b.
c. See figure.
(It does not need to be exact, that is for later.)
d. Choose to which well you go.
e. Yes. See point *X*. The arrows are of equal length and the distances to wells 2, 3 and 4 are larger.
f. Yes. Very far to the southeast.
Figure 3



g. No. Not for 3 and 4. (It does hold for the extensions of the edge.)
h. Straight. (This will be investigated thoroughly later.)

The goal of the authors of the book *Geometry with Applications and Proofs* is to enable students to build a mathematical Voronoi diagram on the basis of the task above and the tasks below. A Voronoi diagram of a collection of points S is a partition of an area into cells, each of which consists of the points closer to one particular point than to any others.

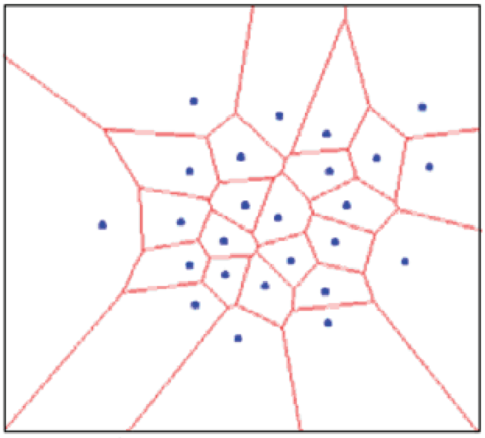


Figure 4. Voronoi diagram

Voronoi diagrams are part of what is known as discrete geometry. Situations like an archeologist attempting to identify the parts of a region under the influence of different Neolithic clans, a meteorologist estimating precipitation at a gauge which has failed to operate, an urban planner locating public school in a city, a physiologist examining capillary supply to muscle tissue, would appear to have little in common. However, all of these problems, together with many others, can be solved by approaches from a single concept. This concept is a simple but intuitively appealing one. Given a finite set of distinct, isolated points in a continuous space, we associate all locations in the space with the closest member of the point set. The result of this procedure is a partitioning of the space into a set of regions, it is known as Voronoi Diagram.

The authors of the book propose the introduction of elements of discrete mathematics in the syllabus of non-university teaching.[1] The method is as follows. Teachers set a series of problems which students must try to solve; they must also justify their responses (normally in groups). Particularly important aspects of their performance are their examination of the task, their justification of the possible answers, their representation, their reappraisal of the solution given, and the arguments they use. As the solutions are pooled together, as well as solving the problems, the unifying concepts are gradually constructed. These concepts are related to each other and are applied to exercises, and are then used to solve more complex contextualized problems.

This book assigns a major role to contextualized problem situations and clearly aims to generate new mathematical objects. The project carried out at the Freudenthal Institute "Realistic Mathematics Education" (Gravemeijer, 1994; De Lange, 1996) proposes a focus on mathematics teaching and learning which conceives the discipline as a human activity like any other, and therefore considers that "knowing mathematics" is the same as "doing mathematics;" as a result, the solution of "realistic problems" should

[1] The discrete algorithms used in computer sciences and computer-based modeling of various phenomena have shifted the emphasis in mathematics today towards discrete mathematics. This has led to calls for its incorporation in mathematics syllabuses. It has even been suggested that certain parts of discrete mathematics are sufficiently elementary to be included successfully in non-university teaching.

be an important part of its study. One of its basic principles is that, for a mathematical activity to be significant, it must depart from the real experience of students (Freudenthal, 1983). Other important principles are that students must be given an opportunity to reinvent mathematical concepts, and that the teaching-learning process must be highly interactive. According to De Lange (1996), there are basically four reasons for including contextualized problems in the syllabus: (a) they facilitate the learning of mathematics; (b) they develop competences; (c) they develop competences and general attitudes associated with problem-solving; and (d) they allow students to see the utility of mathematics for solving situations in other areas such as everyday situations.

Today mathematics is seen as a science in which method clearly predominates over content. For this reason, great importance is given to the study of mathematical processes, particularly the mega-processes "problem-solving" and "modeling." The desert task we discuss here is the first in a series of activities that aim to teach the modeling process. The modeling process is normally considered to follow the five following stages: 1) observation of the situation; 2) simplified description of the situation; 3) construction of a model; 4) mathematical work with the model; and 5) interpretation of results in the situation.

The modeling or mathematization process can also be understood as the result of two other processes: *horizontal* mathematization and *vertical* mathematization. Horizontal mathematization leads from the real world to the world of symbols, and makes it possible to treat a set of problems mathematically. Vertical mathematization is the specifically mathematical treatment of situations.

3. TASK ANALYSIS

In line with the onto-semiotic approach, here student practice is understood as the reading of the task and its subsequent resolution.

In order to carry out a mathematical practice, an agent must have basic knowledge, both to carry out the practice and to interpret the results as satisfactory. If we consider the components of the knowledge that the agent must have in order to develop and evaluate the practice that permits solving a problem (e.g. propose and solve a system of two equations with two unknowns), we can see that a certain verbal (e.g. solution) and symbolic (e.g. x) language must be used. This language is the ostensive part of a series of *concepts* (e.g. equation), *propositions* (e.g. if the same term is added to the two sides of an equation, an equivalent equation is obtained) and *procedures* (e.g. solution by substitution) that will be used in making *arguments* to decide if the simple actions that make up the practice, which is understood to be a compound action, are satisfactory. We will then consider that when an agent carries out and evaluates a mathematical practice, it is necessary that it activates some of the elements mentioned above (or all of them): situation-problems, language, concepts, propositions, procedures and arguments. By articulating these types of objects, we obtain the configuration in Figure 2 (hexagon in Figure 1). The hexagon in Figure 1, referred to as the "epistemic configuration" in the EOS (Figure 5), is a tool that allows us to see the structure of the objects that facilitate the practice that hypothetical students will have to undertake according to the solution envisaged by the authors (for each specific student, we will have a different cognitive configuration). Below we will apply this tool to see which of the objects are active in the hypothetical resolution of the task:

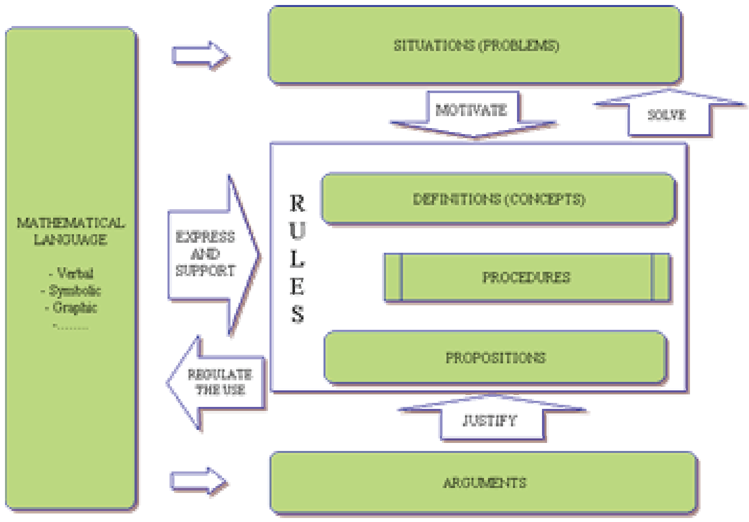


Figure 5. Epistemic configuration

3.1 EPISTEMIC CONFIGURATION OF THE TASK

Situation-Problem: The "In the desert" task – given there are 5 wells, the map has to be divided into five parts in such a way that each contains a well, and so that any place within each domain is closest to this particular well.

Language:[2] (1) terms and expressions: map legend, scale, map, point, region, area, far apart, near, nearest, furthest, division of a region, contains, domain, border, equal distances, larger distances, middle, line segment, cut, intersect, straight lines curves; (2) Graphic representation; (3) Symbolic representation: *J*, 1, 2, 3, 4 and 5.

Map.

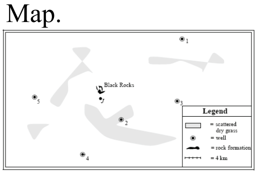


Figure 6.

Sketch showing the division into domains:

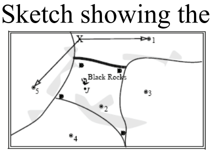


Figure 7.

[2] The sentences are considered representations of definitions, procedures, properties and arguments, and are not included in the "language" category because the definitions, procedures, properties and arguments represented by these sentences appear in the following categories of the epistemic configuration.

Concepts (Definitions): (1) previous or implicit concepts: line segment, end of a segment, greater and lesser (order relations), distance, intersection, belonging to a set, area surrounding a point, border, enclosed domain, unenclosed domain, mid-point, straight line, curve; (2) emerging concepts: a) the domain of a point is formed by the set of points in the plane so that their distance to this point is less than their distance to any of the other points; b) nearest neighbor.

Properties: a) given point *J* and an additional two points, then the point nearest to *J* is that which forms the shortest segment; b) the joining of all the parts is the whole; c) two non-neighboring domains have an empty intersection; and d) the points on the border are equidistant from the centers of the domain.

Procedures: a) estimation of lengths; b) sketch of the division of a region; and c) use of instruments including ruler and compass

Arguments: a) I would go to well 2 as it is the nearest; b) using a compass we can find the points asked for; c) graphic construction; d) I would go to either of the two wells given that they are located at the same distance; e) No, for example the domain of wells 3 and 5; f) Yes, if we extend the line map further south; g) No, for wells 3 and 4; h) They are straight lines.

3.2 PROCESSES

The task given to students is an extra-mathematical situation the resolution of which allows the emergence of, among other things, a new mathematical object: the partition of an area according to the *nearest-neighbor-principle.* The authors intend to present a situation of an extra-mathematical context which is understood by the student as a particular case of a mathematical object. In this case, the particular is extra-mathematical and the general is a mathematical object.

The detailed analysis of the activity needed to solve the task shows that many of the processes in Figure 1 are put into play. The task we are analyzing is divided explicitly into two parts: 1) the statement of the problem; and 2) the commentary that follows the problem. If we look at this from the perspective of the processes considered in the onto-semiotic approach, the statement of the problem aims to generate a process of personalization (in the sense that students construct, among other things, a mathematical object "partition of an area according to the nearest-neighbor-principle"). On the other hand, the subsequent commentary tries to institutionalize this mathematical object, in the sense that it is something known by all the class, that is to say, it comes to "exist" as a mathematical object in the classroom.

Commentary

In the commentary that follows the problem, in seeking to achieve this institutionalization, the authors first generate a hidden process of idealization (the desert becomes an area) and then they establish the particular-general relationship between the idealized situation and the mathematical object. "In this exercise you just partitioned an area according to the nearest-neighbor-principle." The way of presenting the general to students is the result of an additive abstraction, as it consists of a coming together of different elements in the same group. "Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few." We then go on to a process of particularization by saying, "Next we will investigate the simple case of two wells," and one of explicit idealization, when the wells are converted into points "or actually two points, since we might not be dealing with *wells* in other applications."

The statement of the problem

From the outset we assume that hypothetical students can understand the wording of the problem. In other words, we assume an initial process of "communication" (in the sense that "she understands the wording used by others in presenting mathematical problems"), and we shall not analyze this any further. Therefore, we assume that hypothetical students understand what is signified (the process of signifying) in the cartographic representation and its legend, the terms that appear on the map and, above all, that she understands the overall text. This supposition implies that we will not analyze with detail the process of signifying.[3] Moreover, we suppose that students read initially all the items of the text and they are to solve them point by point.

Question a:

"To which well would you go for water? That choice was not difficult. Of course you would go to the nearest well."

Students have to undertake the process of "communication" (understand the task). To respond to this question, students are expected only to undertake a process of "enunciating" a statement or, more specifically, they are expected to respond "I would go to well 2." To be able to give this response, students, in our opinion, have to put into operation, both explicitly and implicitly, certain concepts (line segment, end of a segment, greater and less distance); certain properties (given point J and an additional two points, the point closest to *J* is the one that forms the shortest segment); certain procedures (estimation of lengths) and an argument (thesis: well 2 is the nearest well to J. Argument: Point 2 is the one that is closest to J as can be observed by inspection). We observe that the authors of the book do not consider important that students justify their choice of well 2. Perhaps they consider the answer evident and of course they use the compass or a graduate ruler.

Question b

"Point out two other places from where you would also go to well 2. Choose them far apart from each other."

Students have to undertake the process of "communication" (understand the task). In this question, students are expected to respond simply by undertaking a process of "representation" and "materialization." Specifically, they have to make an ostensive representation such as the following:

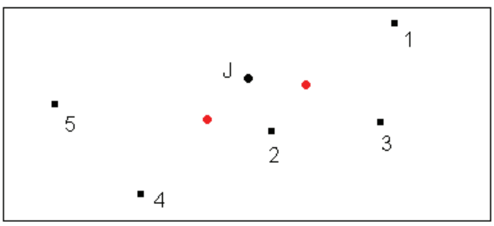


Figure 8.

[3] An example of how the process of signifying would be analyzed is the following contribution to the Topic Group 27 "Mathematical Knowledge for teaching," México: Godino, J. D.; Rivas, M.; Castro W. F. y Konic, P. (2008); "Epistemic and Cognitive Analysis of an Arithmetic–Algebraic Problem Solution," http://tsg.icme11.org/document/get/391

In order to do this, they must formulate an explicit or implicit argument based on: (1) a visual estimation of the fact that the distance that separates the red points from well 2 is smaller than that which separates them from the other wells; and (2) that if the distance between the red points were greater, then there would be another well, other than number 2, that would lie closer to one of the two red points. It is surprising that the authors do not ask students to make explicit the arguments for justifying their representation, given that students are assumed to have this previous knowledge and so they could give this response without any difficulty. For example, students can produce as answer also a sentence such as "each pair of points belonging to a circle centered in well 2 with a radius smaller than the minimum of the distance between the wells no. 3 and no. 4" and they can support this argumentation with a representation where the circle appears.

Question c

"Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest."

Students have to undertake the process of "communication" (understand the task, although in this instance it is not so obvious that the understanding can be assumed). In this question students are expected to respond basically by undertaking a process of "representation" and "materialization." Specifically, they are expected to make an ostensive representation such as the following:

(a)

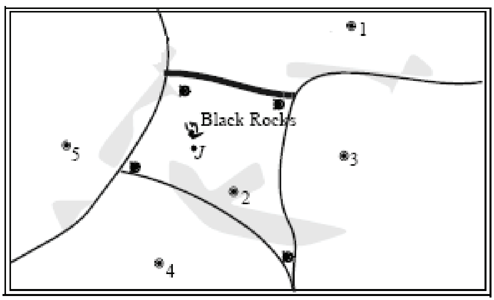
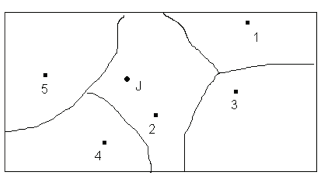
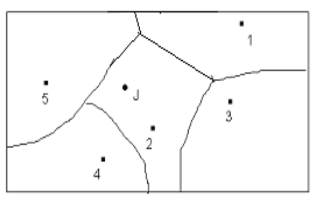


Figure 9.

However, students may offer other valid ostensive representations such as:

(b) (c)





Figures 10 and 11

As a consequence of (1) students' exploration; and (2) the interaction with teachers and other students, they can refine their initial partition (for example, c), giving a strip of different representations supported by their new considerations through which they arrived at their last representation (for example, a).

To be able to provide any of these representations, students, in our opinion, have to put into operation, both explicitly and implicitly, certain concepts (intersection, belonging to a set, area surrounding a point, border) and a process of "conceptualization" (process of implicit definition) must occur that allows the emergence of a new term (domain) and a new concept: the concept of domain (given a set of points on a plane, the domain of a point is formed by the set of points in the plane so that their distance to this point is less than their distance to any of the other points); certain properties (the joining of all the parts is the whole, two non-neighboring domains have an empty intersection, the points on the border lie at the same distance from the two wells); certain procedures (sketch a division of a region in the plane) and an argument (trial and error?).

Question d
"What can you do when you are standing exactly on the edge of two different domains?"

In the previous question students, at least implicitly, had to use the property that states that the points on the border lie at the same distance from the two wells. This question seeks to produce a process of "conceptualization" (process of implicit definition) that allows the emergence of a new term (edge/border) and a new concept: the concept of a border understood as the line that separates two adjoining domains and one of its properties (the points on the border lie at the same distance from the two wells). In this case also the refinement of previous rough representations can be made, where the points of borders verify the equidistance between wells.

Question e
"Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells."

Students' answer here depends on their answer to question *c*. For example, if their answer to question *c* had been representation *a* or *c*, then we would expect them to respond to question *e* by stating that the domains of wells 1 and 5 adjoin and in locating a point that is equidistant from the two wells they would indicate a point on the border, and that this would be located further away from all the other wells. This process of representation of point *X* also requires a process of idealization of the sketch made. Students have to understand that, even if the border that separates wells 1 and 5 had fallen outside of their drawing, at exactly the same distance from the two wells, ideally this is so.

However, if their answer is representation b, then we would expect them to respond that they do not adjoin and that to find a point that is equidistant from the two wells, they would use any of the procedures they know for finding a point that is equidistant from two other points (for example, by drawing a point on the perpendicular bisector, or by trial and error using a ruler and a compass). Given that these points lie outside the figure (as they have to be located further away from all the other wells), they would have to undertake a process of idealization that would lead them to understand that the area that they have to divide is the whole of the plane and it is to be hoped that they would make a new ostensive representation along the following lines:

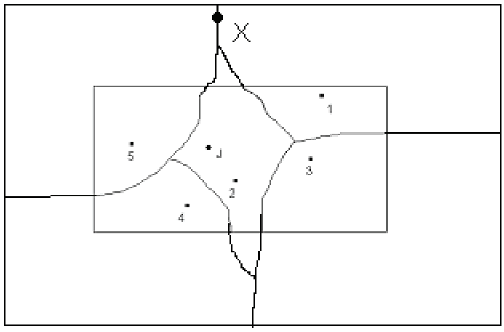


Figure 12.

An alternative answer is suggested in the solutions proposed by the authors. It is expected that students can find with a certain degree of precision a point "*X* which has equal distances to wells 1 and 5 and has larger distances to all the other wells" and that this point might not be on the border of the division sketched by students (since this is no more than a sketch). In this case a semiotic conflict would occur that could only be resolved by resorting to a process of idealization of the division sketched which would allow students to suppose that *X* would be a point on the border if the partition had been carried out accurately.

Question f

"In reality the desert is much larger than shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?"

Students have to undertake the process of "communication" of the task (we assume that hypothetical students can understand the wording of the problem). To respond to this question students are expected only to undertake a process of "enunciating" a statement or, more specifically, they are expected to respond "Yes, very far to the southeast." But in order to give this response, we believe that students should have undertaken a process of idealization (as the question suggests) that leads them to extend the plane; and a process of ostensive or non-ostensive representation, to indicate the border that is not shown in the sketch.

In our opinion, students who answered question *e* based on representation *b* would almost certainly have reached the answer to this question on their own.

Question g

"The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?"

Students have to undertake the process of "understanding" the task (we assume that hypothetical students have the ability to do so). To respond to this question students is expected to undertake a process of "enunciating" a statement – specifically they are expected to respond "No," and a process of argumentation, giving a counterexample such as "This is not the case for wells 3 and 4." In order to give this answer, we believe that students have to inspect visually the border that separates two wells and make a non ostensive (or equally ostensive) representation of the midpoint for the extremes of the two wells in deciding whether the border passes through the midpoint. If it does, they must continue until they find a counterexample. To do all this, they have to put into operation, both explicitly and implicitly, certain concepts (midpoint of a segment and the intersection of curves) and procedures (visual estimation of the midpoint and point of intersection).

Question h

"What kind of lines are the edges? Straight? Curved?"

Students have to undertake the process of "understanding" the task (we assume that hypothetical students have the ability to do so). To respond to this question students are expected to undertake a process of "enunciating" a conjecture (which might be either "straight" or "curved"). The authors suggest that teachers tell students that the correct conjecture is "straight" and that this will be justified at a later date. It is not entirely clear why this question is included when the reasons for the validity of the conjecture are omitted.

4. CONCLUSION

In this paper we have shown how the use of the "epistemic configuration of mathematical objects" construct, together with the processes considered in the onto-semiotic approach, allows us to undertake a better analysis of mathematical tasks and practices, one of the skills needed to analyze our own professional practice. The "epistemic configuration" tool proves useful for the static description of the structure (organization, configuration, anatomy, etc.) of a mathematical text, while the processes are tools that enable us to explore more thoroughly the operation (dynamics, physiology, etc.) of the epistemic configuration activated in the realization of the mathematical practice. Therefore, we show how the onto-semiotic approach can help us analyze mathematical texts and thus help us avoid stumbling blocks and understand students' conceptual problems.

As Hiebert, Morris & Glass (2003) affirm, a persistent problem in mathematics education is how to design educational programs that influence the nature and quality of teachers' practices. Tools for analyzing the educational practices, such as those proposed here, are necessary for designing these programs, as they provide a structured opportunity for reflection.

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