Bolite Frant, J.; Acevedo, J.I.; Font, V. (2006). metaphors in mathematics classrooms: analyzing the dynamic process of teaching and learning of graph functions. *Actas del CERME 4* (pp. 82-91)

METAPHORS IN MATHEMATICS CLASSROOMS: ANALYZING THE DYNAMIC PROCESS OF TEACHING AND LEARNING OF GRAPH FUNCTIONS.

Janete Bolite Frant *PUC-SP*; **Jorge Acevedo** *UB &***Vicenç Font** *UB*

Abstract: *The purpose of this paper is to analyse a phenomenon that is observed in the dynamic process of teaching and learning of graph functions in high school*[1]*: the teacher uses expressions that suggest, among other ideas, (1) orientation metaphors, such as "the abscise axis is horizontal", (2) fictive motion, such as "the graph of a function can be considered as the trace of a point that moves over the graph", (3) ontological metaphors and (4) conceptual blendings.*

1 INTRODUCTION

In this research we have tried to answer the following four questions: What type of metaphors does the teacher use to explain the graphic representation of functions in the high school? Is the teacher aware of the use he/she has made of metaphors in his/her speech and to what extent does he/she monitor them? What effect do these metaphors have on students? What is the role played by metaphors in the negotiation of meaning?

This paper is divided into five sections. The first section contains an introduction and comments on the research problem. The second section reviews the research on metaphor and presents the theoretical frameworks of embodied cognition. The third section presents the study and its methodology. The fourth section contains the data analysis and our answer to the four questions that are the goal of the research. Finally, in section five, we offer some conclusions.

2. BACKGROUND

In recent years, several authors (e.g., Font & Acevedo 2003; Johnson, 1987; Lakoff & Núñez, 2000; Leino & Drakenberg, 1993; Núñez, 2000, Presmeg, 1992, 1997; Sfard, 1994, 1997) have pointed out the important role played by metaphors in the learning and teaching of mathematics.

We start by considering metaphor as an understanding of one domain in terms of another. According to Lakoff and Núñez (2000), metaphors generate a conceptual relationship between a source domain and a target domain by mapping and preserving inferences from the source to the target domain. Because metaphors link different senses, they are essential for people in building meanings for mathematical entities *"…a large number of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical in nature"* (Lakoff and Núñez p. 364). However, not all conceptual mappings draw from direct physical experience, or are concerned with the manipulation of physical objects. We are also aware that only some aspects of the source domain are revealed by a metaphor and in general, we do not know which aspects on the source domain are mapped by the students.

[1] Bachillerato in Spain

Although conceptual metaphor is directly related to the person building it, in classrooms, teachers use a metaphor, consciously or otherwise, to try to explain a mathematical subject to students more clearly, i.e., in order to facilitate students' understanding. We investigate the implications of this practice for students' understanding of mathematics.

3 METHODOLOGY

The research presented here is a theoretical reflection based on analysis of various teaching processes for the graphic representation of functions in the Spanish high school diploma. The classroom episodes and interviews mentioned in this paper are part of the field material used as the basis for the reflections and results shown here.

The information was obtained at the place of work of the subjects researched. The teachers who participated in this research did so voluntarily and gave their specific consent to interference with their teaching work (class observations, video recording, analysis of working materials, etc.). The students participated at the teacher's request. The choice of the teachers and students recorded on video was not made based on any statistical criterion. Only their willingness to co-operate and to be recorded was taken into consideration.

In this paper, we are going to look especially at the recording of the classroom sessions of teacher A. Two other teachers (B and C) are also referred to, as is the interview, recorded on video, with a student of teacher C, who we will refer to as student D.

In order to analyse the teachers' teaching processes effectively, we need written texts. For this reason, we videotaped his lessons and transcribed them. We organised the transcription into three columns. These were (1) transcriptions of the teacher's and students' oral discourse, (2) The blackboard and (3) comments on the teacher's gestures. Our focus was on the teacher's discourse and practice, so the students' discourse and practice[2] appears only when interacting with the teacher.

Once we had these written texts, we needed to separate them into analysis units. One possible way to perform this separation was to take the construct "didactic configuration" as the basic analysis unit. Godino, Contreras and Font (2004) consider that a *didactic configuration* – hereinafter referred to as a DC – is established by the teacher-student interactions based around a mathematical task.

The teaching process for a mathematical subject or contents takes place in a timeframe by means of a sequence of didactic configurations. Although the basic

[2] We feel that mathematics learning means becoming able to carry out a practice, and above all, to perform a discursive reflection on it that would be recognised as mathematical by expert interlocutors. From this perspective, we see the teacher's speech as a component of his professional practice. The objective of this practice is to generate a type of practice within the student, and above all, a discursive reflection on it, which can be considered as mathematics.

criterion for determining a DC is the performing of a task, grouping in didactic configurations is flexible and at the researcher's discretion.

Analysis of the didactic configurations implemented in a teaching process is facilitated if we have some theoretical models for use as reference. Godino, Contreras and Font (2004) mention four types of theoretical configurations that can play this role and which are designated as *teacher-centred*, *adidactic*, *personal* and *dialogue-based* configurations. The empirical didactic configurations that arise in the teaching processes carried out are indeed close to one of these four theoretical configurations.[3].

Division of the classroom session into didactic configurations enables subsequent macroscopic analysis of a wide range of didactic configurations, while finer (microscopic) analysis will be carried out mainly on a much smaller number of these didactic configurations. In our research, after defining a DC, we focused our analysis on the phenomena related to the use of metaphors seen in it.

4 DATA ANALYSIS

In this section, we will perform the data analysis and answer the four questions that are the objective of the research

4.1 Reply to the first three questions

As far as the first question is concerned, the use of *orientation metaphors* can be seen in the teachers' explanations. For example, we can see that teacher A is using "horizontal" instead of saying "parallel to the abscises axis", "horizontal axis" instead of "abscises axis" and "vertical axis" instead of "ordinates axis". This is stressed not only in his speech, but also in his gestures. Only in one DC did the teacher fail to identify the ordinates axis as the vertical axis and the abscises axis as the horizontal axis although interestingly, the text book never made this identification.

Teacher A: …in x =0 shows a minimum and the derivative in x = o is zero as we could expect, because now the tangent line is horizontal…[While he says this, he gestures with his hands, indicating the horizontal position of the tangent line on the graph on the blackboard]

We can find also metaphors which facilitate students' understanding of the idea that "*the graph of a function can be considered as the trace of a point that moves over the graph*".

Teacher A: …if before 0 is increasing, if after 0 is increasing, if before 0 and after 0 is increasing we have an inflexion point. If before 0 is increasing and after 0 is decreasing, it's a maximum. If before 0 is decreasing and after 0 is increasing, a minimum. [Gesturing comes along these comments in the graph of the blackboard].

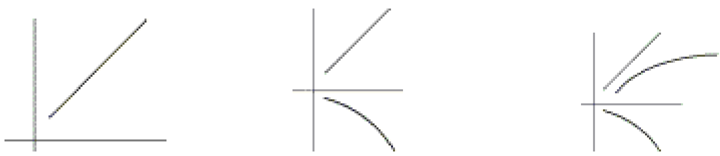
[3] The lack of adidactic DCs and the presence of some dialogue-based DCs in the classroom sessions recorded on video, seems to suggest that they are quite similar to the traditional mathematical classroom - featuring one blackboard, one teacher as the focus of discussion and twenty to thirty silent students which seems to belong to history.

In the teacher's discourse, we find a powerful metaphor, *the fictive motion* (Lakoff and Númez 2000). He, teacher A, uses expressions like "before 0" and "after 0" in such a way that the point 0 is understood as a location determined on a path (function). According to the authors, this is ubiquitous in mathematical thought (p. 38). There is a spatial organisation, suggesting an origin (from), a path (where the function goes) and a goal (to, until). The essential elements in this schema: are a trajectory that moves, a route from the source to the goal, the position of the trajectory at a given time.

Font (2000) and Bolite Frant et al. (2004) found that when teachers explained a graph of a curve as the trajectory of a point that moves, the students thought point *A* would be the same after being moved, as when a person or a car moves from one place to another in space, they are still the same person or car. Here we see that for the teacher, only part of a source domain from daily life (things moving in space) was mapped, while the students were mapping a bigger scene. In other words, teacher has a clear idea of what features were to be mapped while the students do not.

Another type of metaphor observed are *ontological* - which enable events, activities, emotions, ideas, etc. to be considered as if they were entities (objects, things, etc.) - and *metaphorical blends*. For example, a mixture of ontological and dynamic metaphors can be seen in the following transcription from teacher A.

> *Teacher A:* One of the things we study to representing the graph of a function is the behavior at the infinity. What does the function do when *x* tends to infinity? What does the graphic of a function do when *x* tends to infinity? It could do this, going towards positive infinity [while drawing the left-hand graph]. It could do this, going towards negative infinity [he draws the centre graph on the previous graph]. It could also increase and stabilise until a certain number, like this [he draws the right-hand graph over the graph in the centre. In the three graphs the teacher moves his hand, making movements that are a continuation of the part of the graph drawn, suggesting an indefinite continuation].



In order to answer the second question, a semi-structured interview with the teachers took place, which was also video recorded. The teachers' level of awareness of their use of dynamic metaphors and their possible effect on students' understanding differs from teacher to teacher. The teacher who gave the class we have used so far, teacher A, was more aware than others. However, Font and Acevedo (2003) consider the case of teacher B, and it can be seen that he is not aware that he uses dynamic metaphors and, therefore does not control them. As a consequence of the interviewer's questions, teacher B realises that he uses them, but feels that this use facilitates understanding and does not feel that the possible difficulties that they may cause his students are important. In fact, he feels that the use of metaphors does not lead to any type of conceptual error among his students.

In order to answer the third question, various students were interviewed and recorded on video, questionnaires were also given to some students and some of the students' productions during the teaching process (for example, examinations) were analysed. A significant example is the case of one of teacher C’s students, who had a good command of the graphic representation of functions. This student was asked to comment verbally on the prior steps (domain; cuts with axes; asymptotes and behaviour at the infinity; study of maximums, minimums, increasing or decreasing intervals; study of inflection points and concavity and convexity intervals) and construction of the graph in the examination. Both the graph and the steps prior to his examination answer were correct.

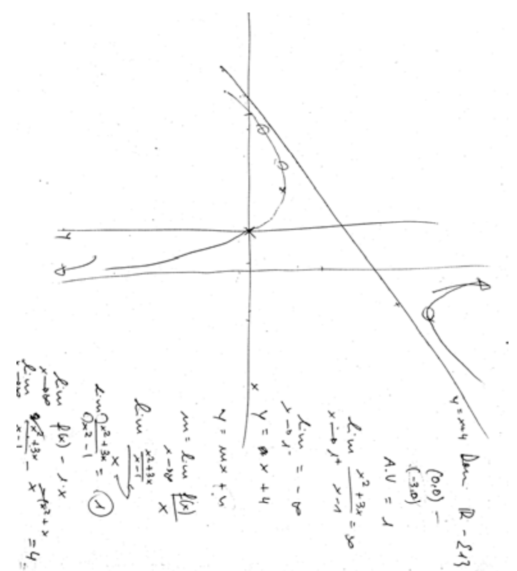
While no metaphor was observed in his written answer, they were omnipresent in his explanation of how he had constructed the graph. For example, in response to the question "Can you now tell me when the function will be increasing and when it will be decreasing?" the student correctly answered by pointing to the intervals and saying that “it increases here because it goes up and decreases here because it goes down.”)

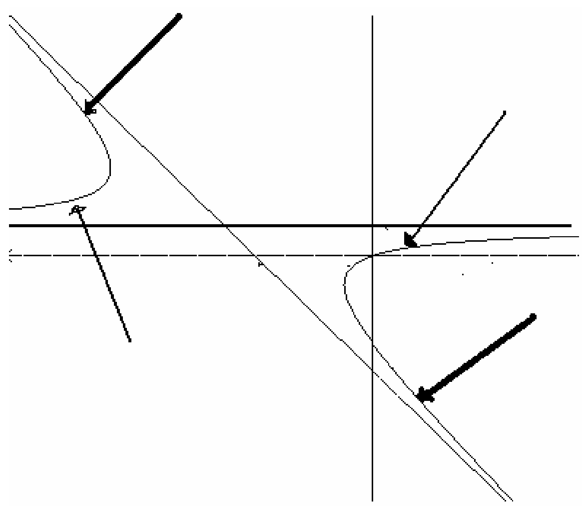
Interviewer: Can you now tell me when the function will be increasing and when it will be decreasing? [While putting the paper on which the student has drawn the graph of the function in its horizontal position].

Student D: [Hesitates for a few seconds] I don't understand, do you mean that the axes have changed?

Interviewer: No, the axes haven't changed, they're still the same.

Student D: This one is decreasing because it is going down and this one is increasing because it is going up, this other one is decreasing because it is going down and this one is increasing because it is going up. [He hesitates for a few seconds and points to the part of the curve shown with a thin arrow as increasing and that shown with a thick arrow as decreasing]

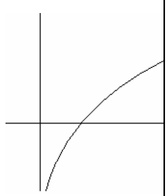




4.2 Metaphor and Meaning Negotiation

We now see an example of the role played by metaphors in the negotiation of meanings, which is understood as the connection between personal and institutional meanings in a teaching process.

The division of teacher A's classroom session into DCs enabled one to be determined which begins when the teacher suggests the task of calculating the domain of a function and ends when the teacher proposes two new tasks. (First he tells the students to solve an activity based on calculating domains at home, and then suggests finding the points where a function cuts across the axes in class). In this DC, teacher A wanted to recall the "domain of a function" and the techniques used to determine it, which had been studied beforehand, and he used three examples. This is a teacher-centred type DC with an attempt by the teacher to make it dialogue-based.

Transcripts of the DC	Blackboard	Notes
T: So let's start with the domain. Remember that the domain of a function is the set of values of the independent variable that has an image. …… Or to put it another way; they are the values for which I can find the image, they are the *x* where I can calculate the image. For example, look at this function $f(x)=1/(x+1)$. The domain of this function consists of the set of numbers for which when I substitute the *x* for these numbers I can carry out this entire calculation, that is, I can find the image.	$f(x)=1/(x+1)$	He points to the *x* of the formula. He moves his hand around the fraction $1/(x+1)$.
T: Can this always be done? Except for one number, which one? S: -1 T: Then the domain is real numbers except for -1, that is, you can find an image for any number except for -1		
T: There are more complicated functions, such as the neperian logarithm of *x*, for example.	$f(x)=\ln x$	Teacher writes on the blackboard "D(f)=0,+∞)". He points to zero with the fingers.
T: What is the domain of this function? Think about the graph and from there.... Tell me. S: From zero to positive infinity. T. Yes, from zero towards positive infinity is the domain, because logarithms of negative numbers do not exist, the logarithm of minus one does not exist. Is zero included or not included?	D(f) = 0,+ ∞) 	The teacher draws the graph and points to it. with the hand The teacher gestures with his hands following the line of the

S: No T: No …very good... So the domain of this function is from zero towards positive infinity. Remember that the graph of this function, did something like this,.. The graph of this function did something like this, and the domain is from zero towards positive infinity. T. Any doubts? T: A final example, the square root of *x*, What is the domain of this function? ….Come on!! S:…(inaudible, but it is an incorrect answer) T: Ah yes! T: Except for the negatives … because the square root of a negative number does not exist, we could also say the same real numbers except for the negatives, easier, all the positive numbers, we can put it like that, easier, we can express it in the form of an interval, from zero to infinity, zero is included this time, it is included.	D(f)=(0,+∞) $f(x)=\sqrt{x}$ D(f)=[0,+ ∞)	graph. Then he points to the zero, and moves it towards the right to represent the interval (0,+ ∞). Teacher writes on the blackboard "(" before the zero

First the teacher introduces the formulation "the domain is the set of values of the independent variable that has an image". Then he continues: "they are the values from which I can find the image". The second remark is more functional in finding the domain than the first; since it facilitates a "language game" that allows a common meaning about which the domain in question is. The characteristics of this "language game" for the function $f(x)=1/(x+1)$ are: 1) *Introduction of a generic element*. The teacher introduces the element *x* which allows operation of the function formula according to "when I substitute the *x* (his finger is on the *x* in the given formula) for these numbers I can carry out this entire calculation (with his hands surrounding the fraction 1/(x+1)), that is, I can find the image". Then he waits for the students to mentally find the values for which the operations indicated in the formula of the function cannot be carried out. 2) Agreement of the range of values of the generic element. The students raise some hypotheses about the domain until they came to an agreement that was accepted by all, including the teacher. Several students say "-1" and the teacher is satisfied with this answer.

In the function $f(x) = \ln x$, the same language game is reproduced, with certain differences. The first is that the generic element is a point in the negative part of the abscises axis. The teacher draws the graph of $f(x) = \ln x$ and waits for the students to mentally apply the following technique: (1), thinking of a negative point; (2) tracing a line perpendicular to the abscises axis passing through this point; (3) observing that this line does not cut the graph of the neperian logarithmic function and, (4) stating that this reasoning is valid for any negative point and also for a point in the origin (this technique was shown in a previous unit). The second difference is that, when the students answer “from zero to positive infinity” the teacher considers it to be ambiguous and decides to intervene, asking them if zero is a point of the domain; he then accepts the students’ answer that zero is not the domain.

It is important to note that both answers is expressed in metaphorical terms. Students and teachers use the expression “from zero to positive infinity”. The students do so orally and the teacher adds a written expression $(0,+\infty)$ and gestures towards the positive part of the abscises axis (moving his hand from the origin to the right. This is the metaphor that considers the semi-line number as a path with a source (start point) and a goal (positive infinity).

The synchronism of dynamic language and hand movement allows students to understand the domain, a case of actual infinity, since it is an open interval, as the result of a movement that has a beginning but no end. According to Lakoff and Núñez (2000 p.158), we see this case of actual infinity as the result of a movement that has a beginning and no end, due to the fact that we metaphorically apply our knowledge of processes which have a beginning and an end to this type of process. This is what these authors call the BMI – the Basic Metaphor of Infinity.

5 FINAL CONSIDERATIONS

This paper revealed that conceptual metaphors are relevant tools for analysing and improved understanding of mathematics classroom discourse. In one way it is already embedded in theoretical concepts -e.g. the values above the origin (the ordinates axis) are positive. In the other, it is present in teacher’s explanation when for in order to facilitation purposes, in order to turn theoretical concepts into intuitive ones, he used metaphors that may relate directly to students’ daily experience - e.g. the vertical axis as the ordinates axis. It is also present in the way students organise their knowledge – e.g. of the Cartesian axis based on spatial orientation based on their bodies.

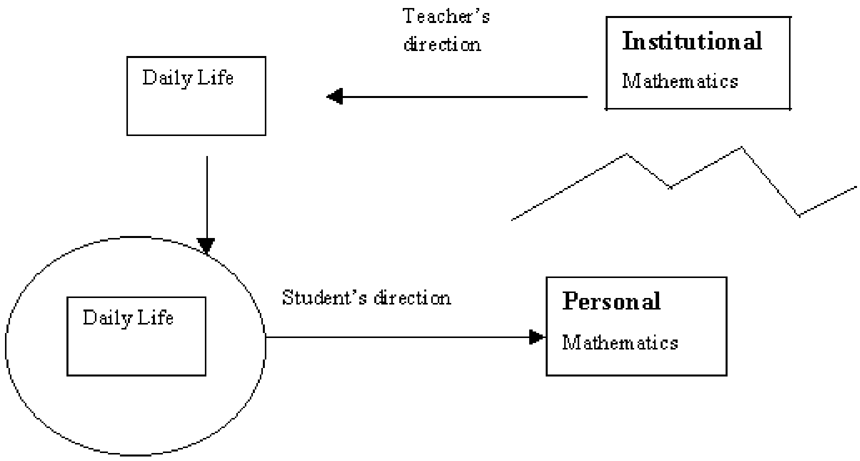
We found that the use of several metaphors (orientational, fictive motion, ontology, and metaphorical blends) is present in both the teacher’s and students’ speech. This is inevitable and sometimes unconscious, but it is fundamental in building/talking mathematical objects.

As well as a description in global terms, the graphic representation of functions also requires the introduction of local concepts such as increasing and decreasing at a point, etc. formulated precisely in static terms, using the notion of number sets. These local concepts are very difficult for high school students, and for this reason many teachers leave them in the background and prefer to use dynamic explanations, in which the use of dynamic metaphors is fundamental, which they consider more intuitive. Students' productions also show that the use of these metaphors in the teacher's speech has significant effects on students' understanding.

Metaphors, as seen here, also play an important role in negotiating meaning in classrooms, and we propose a model that takes the dynamic of the interplay of discourses into account. It is important to note that metaphors in classrooms may have two different directions. On the one hand, there are metaphors that teachers use in the belief that they are facilitating learning, and on the other there are students' metaphors.

The teachers' source domain is mathematics and the target is daily life because they try to think of a common space to communicate with the students. However, the domain of daily life is not always the same for both, because the teacher is using only the part of the daily life concept that is mapped into the mathematical domain. Students usually have a larger daily life domain than that which is mapped and is not in the same mathematical teacher's domain.

The use of metaphors has its advantages and disadvantages. The teacher must therefore make a controlled use of them and must be aware of their importance in students' personal objects.



BIBLIOGRAPHY

Bolite Frant, J. et al. (2004). Reclaiming visualization: when seeing does not imply looking. TSG 28, ICME 10, Denmark [http://www.icme-organisers.dk/tsg28/]

Font, V. (2000). *Procediments per obtenir expressions simbòliques a partir de gràfiques. Aplicacions a les derivades*. Tesis doctoral no publicada. Universitat de Barcelona.

Font, V. & Acevedo, J. I. (2003). Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones. *Enseñanza de las Ciencias*, 21, 3, 405-418.

Godino, J. D., Contreras, A. & Font, V. (2004). *Análisis de procesos de instrucción basado en el enfoque ontológico-semiótico de la cognición matemática*. XX Jornadas del SI-IDM. Madrid 2004.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason.* Chicago: University of Chicago Press.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Leino, A.L. & Drakenberg, M. (1993). *Metaphor: An educational perspective.* Research Bulletin 84, Department of Education, University of Helsinki.

Núñez, R. (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics, en Nakaora T. y Koyama M. (eds.). *Proceedings of PME24* (vol.1, pp. 3-22). Hiroshima: Hiroshima University.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23 (6), 595-610.

Presmeg, N. C. (1997a). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 267-279). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44-54.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339-371). Mahwah, New Jersey: Lawrence Erlbaum Associates.