VISUALIZING AND COMPARING TEACHERS' MATHEMATICAL PRACTICES

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We report the use of a tool to display and analyze essential elements of mathematical activity (definitions, properties, processes, etc.) arising during the development of a class. It has been applied to the study of the commonalities and differences among three classes conducted by three different teachers in the same institution, year and school level when they teach the bisector. The results allow us to infer some aspects about the mathematical knowledge of the teachers involved.

INTRODUCTION

Research on mathematical knowledge and the professional development of teachers has become increasingly important in recent years, and has revealed not only its complexity but also the limitations of the results (Sullivan and Wood, 2008). In particular, we need a research agenda that links theoretical outcomes with practice, conceptualizes teaching practice, and describes and discusses how mathematical knowledge for teaching is developed during a class.

Several authors assume the complexity of mathematical objects and that teachers' mathematical knowledge is large, intricate and evolves constantly. According to Davis and Renert (2013), instead of thinking of teachers' mathematical knowledge of concepts as a discrete set of basic knowledge in the heads of individuals, it is more productive to think of it as systems of changing instantiations (formal definitions, algorithms, metaphors, images, applications, gestures, etc..) emerging in practice and distributed among the whole community of professional teachers. Also, it is important to note that each of these instantiations has different conceptual value, and that it is desirable for teachers to know as many of these instantiations as possible in order to build a rich net of connections between them and acquire a more robust knowledge of the concept. We also assume that the complex nature of mathematical objects may be perceived in different institutions and different historical moments, different textbooks, or different methodological approaches. From this it follows that comparing the practice of different teachers working in the same institution, presenting the same mathematical concept at the same level and at the same time, will enrich our understanding of teachers' mathematical knowledge in practice. From this standpoint, the objectives of this research are as follows:

- To design a visualization tool that displays mathematical knowledge arising during a math class in terms of formal definitions, properties, examples and mathematical processes, among others.

- To use this tool to highlight the essential elements of the mathematical knowledge of the bisector used by the teachers during their practice.
- To account for the commonalities and differences of the mathematical activity of different teachers presenting the same mathematical content (bisector) in the same school year and the same institution.

TOOLS FOR VISUALIZING AND ANALYZING PRACTICE

In order to investigate mathematical practice we need tools specifically designed to address its complexity. To this end research in mathematics education has produced specific methodological tools and analytical frameworks. For example, the work of Rowland and colleagues (Rowland, Huckstep and Thwaites, 2005) and the knowledge quartet provides very specific conceptualization of practice. The works of Tomás Ferreira and Da Ponte (Tomás Ferreira, 2005; Martinho and Ponte, 2009) emphasizing the role of the teacher in the communication process, and the contributions from the Lesson Study methodology (Fernandez and Yoshida, 2004) include a collaborative model to enhance teachers to plan, implement, monitor, and reflect on math classes. All of these are important contributions that allow us to contextualize the interest of our research problem, namely to visualize the essential elements of the mathematical activity arising during classroom practice (definitions, properties, mathematical processes, etc.).

We observed the practice of three different teachers (hereinafter Laura, Antonia and Encarna) when they taught the perpendicular bisector in the final year of primary school (ages 11-12). The three classes were videotaped and transcribed for later analysis, looking for commonalities and differences in the mathematical activity. For the methodological design of the instrument we have decided to use the ontosemiotic approach for three reasons: First, it allows us to describe how mathematical objects emerge in the classroom and pay attention to the complexity associated with the object (*concept study* in the terminology of Davis (2013)). Secondly, because the ontosemiotical approach characterizes mathematical activity in terms of practices, objects and processes when primary and secondary curricula are also structured in terms of processes (reasoning, communication, modeling, etc.). Finally, because the ontosemiotic approach has already yielded visualization tools. Godino, Contreras and Font (2006) made an attempt to display primary mathematical objects during a class, and in this paper we enrich these attempts by incorporating also emergent processes.

For the analysis of transcripts we used one of the levels of the ontosemiotic approach (Godino, Batanero and Font (2007); Pochulu and Font (2011). It focuses on the primary objects and the mathematical processes involved in conducting practices, as well as those emerging from them. For the ontosemiotic approach (hereinafter OSA), mathematical activity is modeled in practices where primary objects (we will refer here to definitions, properties, construction procedures and problems) emerge. On the other hand, instead of giving a general definition of process, the ontosemiotic approach selects a list of processes that are considered important in mathematical activity, without claiming that such a list includes all the processes implicit in all mathematical activities (we will refer here to automation, institucionalization, argumentation, communication, modeling, and connection). Tasks or problems are considered primary objects because they are triggers of the mathematical activity.

An example to distinguish primary objects and processes is the construction of the bisector. For this construction, the student performs a sequence of actions, such as those underlying the use of the ruler and compass. In particular, students can use a construction procedure (algorithm) using only a ruler and a triangle with a 45 degrees angle (it is a procedure, which is considered a primary object type in EOS). With the repetition with other similar exercises the student engages in a process of automation.

TEACHING THE BISECTOR: MATHEMATICAL KNOWLEDGE INVOLVED

Detailed analysis of primary objects and mathematical processes illustrates relevant aspects of the structure and development of each of the three classes and permits us to distinguish many instantiations (using the terminology of Davis and Renert, 2013) of the bisector, as well as to establish relationships between them. Table 1 summarizes objects and processes that emerged from the practice of the three teachers, as well as the codes used to label them in Figures 1,2,and 3.

Table 1: Mathematical objects and processes emerging from the practice.

Mathematical objects	
Definition of perpendicular bisector: Teachers refer explicitly or implicitly to one definition of perpendicular bisector. It includes also definitions of related as segment or line.	
D_1:	Perpendicular line passing through the midpoint of the segment.
D_2:	Locus of all points equidistant from two given points.
D_2A:	Locus of all points equidistant from the ends of the given segment.
D_2B:	Line (boundary) which separates the plane into two regions, so that in a region all the points are nearer one of the two points than the other.
Properties: Any statement regarding the definition and the construction method of the perpendicular bisector, which can be true or false, but there is an attempt to justify it in class.	
P_1:	The point where the line intersects the segment is the midpoint.
P_2:	The obtained line is perpendicular to the given segment.
P_3:	Points on the boundary (D_2B) are aligned.

Construction procedure: Construction algorithm of the perpendicular bisector.	
Pr_1:	Euclid's procedure (Book I prop. X): given a *finite straight line*, describe an equilateral triangle on it (Prop. I) and bisect its angle (prop. IX).
Pr_2:	Perpendicular bisector as a locus: Given two points, find any other two points equidistant from them and connect these last two points with a line.
Pr_3:	Carpenter's procedure: Given a segment, measure its length, take its half and draw the perpendicular at the midpoint with the triangles 45 or 60 degrees, or the protractor.
Problem: tasks that incite mathematical activity, examples and counterexamples.	
EP:	Task based on paradigmatic examples
ENP:	Task based on non paradigmatic examples
CE:	Counterexamples

Mathematical processes		
Institutionalization: A definition, property or procedure is explicitly considered as valid, so from that moment on it is assumed to be known.		
Automation: Students are asked to repeat a certain procedure mechanical and individually.		
Communication: Oral or written statements on mathematical contents are expressed or understood. We explicitly exclude from this category mathematical arguments. Three subcategories have been included:		
EP: Teacher's lecture	DPA: Dialog among teacher and students	DA: Dialog among students
Argumentation: Existence of chains of mathematical arguments		
Modelling: At least one of the following phases of modelling (Blom, 2002) occurs: (a) starting point is a certain situation in the real world; (b) simplify, structuring and making the content precise; (c) objects, data, relations and conditions involved in it are translated into mathematics, and mathematical results derive, (d) retranslation into the real world.		

COMMONALITIES AND DIFFERENCES AMONG THE THREE CLASSES

In each of the three classes there emerge different definitions and construction procedures. In Laura's class she defines the bisector as the perpendicular line passing through the midpoint of the given segment (D_1), and uses Euclid's procedure (PR_1) for the construction. The carpenter's procedure (PR_3), which is suggested by a student, is not institutionalized. In Antonia's class, she institutionalizes the same definition (D_1), but a different procedure (PR_2: given two points, find two equidistant from them and connect them with a line). In Encarna's class, she refers to the bisector as the boundary between two regions in the plane so that in a region all the points are nearer to one of the two points than the other (D_2B), and does not give any construction procedure.

In each of the three cases there appears one process that predominates, occupying approximately three quarters of the total time. In Antonia and Laura’s classes it is the automation process, which takes about the last 75% of the class. In contrast, in the case of Encarna’s the communication process predominates, taking up approximately three quarters of the total. We emphasize that in this third class modeling and argumentation processes appear, which have little or no presence in the other two.

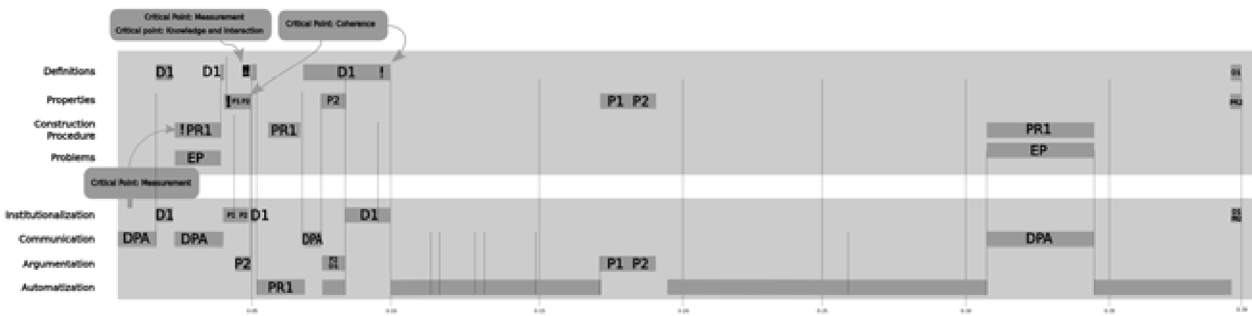

Figure 1. Graphic visualization of Laura's class. See Table 1 for label descriptions.

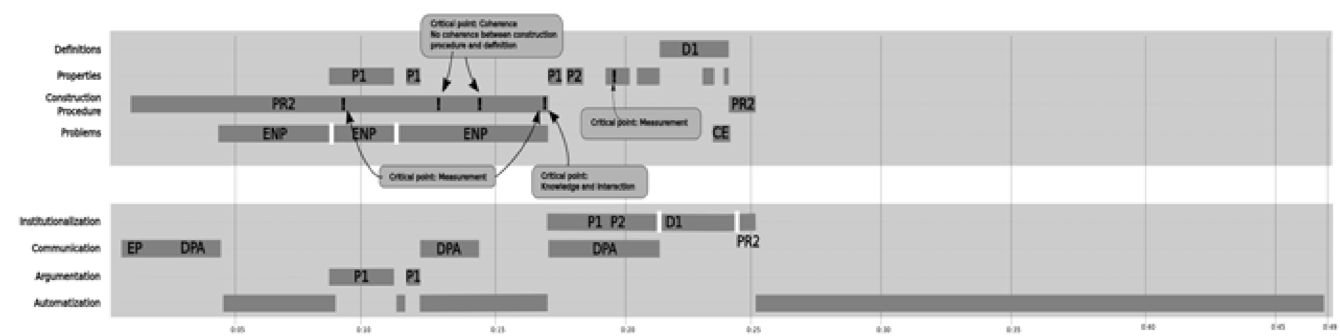

Figure 2. Graphic visualization of Antonia's class. See Table 1 for label descriptions.

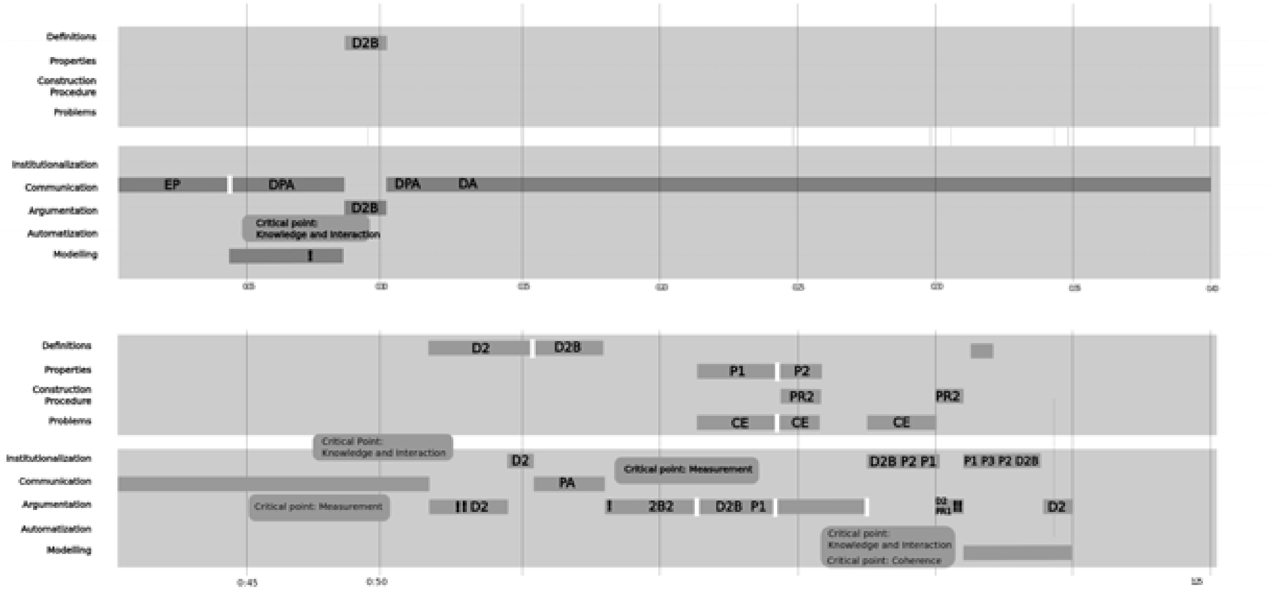

Figure 3. Graphic visualization of Encarna's class (significatively longer). See Table 1 for label descriptions.

A first analytical approach to the graphics shows also that mathematical activity is not uniformly developed during each class, but in all three we find slots of time in which the density of processes, definitions, procedures, etc. is higher. These periods of time occupy approximately from one quarter to one third of the total time of the class. However, in the case of Laura (Figure 1) there is an important accumulation phase during the first ten minutes, when most primary objects are presented. In the case of Antonia (Figure 2) this phase begins approximately in minute ten, while in Encarna's class (which is significantly longer) it appears at the end (Figure 3). We have analyzed in detail the way in which the three teachers connected primary objects and processes (considered instantiations of the bisector) as they emerged in these time slots.

CRITICAL POINTS IN TEACHING THE BISECTOR

The two following extracts from Laura's class illustrate some lack of consistency regarding the use of measure for constructing the bisector or proving its properties. In the first one, a student intends to find the midpoint of the segment using a graduated ruler, but the teacher makes explicit that direct measurement is not permitted for the construction.

Excerpt 1:

Teacher: So the bisector of the segment is nothing else than the straight line perpendicular to this segment that divides exactly it into two equal parts, right? What do you do to get the midpoint of that segment and split it into two equal halves? Say.

Student: I could put this on it -raising a ruler- and measure it.

Teacher: I could measure it with the rule but, would I obtain the same? Exactly? Exactly?

Student: With the compass.

Teacher: [nodding] With the compass (takes the chalkboard compass). The compass is the right tool with which the midpoint of the segment is going to be perfect.

However, in the second extract below, the same teacher uses direct measurement with the angle protractor to verify that the properties of the definition hold.

Excerpt 2:

Teacher: Therefore, one condition is that the line dividing the segment into two equal parts, the bisector of the segment, must be perpendicular. How can I know if these two lines are perpendicular? What do I have to do? Perpendicular (she points the four quadrants in the chalkboard)

Student: Measuring with the protractor.

Teacher: [nodding] Measuring with the protractor. (takes the chalkboard protractor)

Student: A right angle.

Teacher: and I have to obtain...

Students: A right angle, ninety.

Teacher: and I have to obtain four right angles. One, two, three, and four. If I put the protractor here (on the first quadrant)... Let's see. Note that I obtain exactly 90 degrees. OK? And If I put it this way I also obtain 90 degrees exactly. So I can say that the bisector of the segment is the line which is perpendicular to that segment and divides it into two perfectly equal parts. Exactly.

The selected dialogues above reveal as a fundamental aspect of professional knowledge related to the teaching of the perpendicular some reflection on the foundations of the mathematical activity. For the students, finding the midpoint of a segment leads naturally to a problem of direct measurement with the rule, while if the decision of the teacher is to follow the norms of Euclidean geometry, measuring with the ruler has no place in the constructing or proving properties of the bisector. This difference creates a professional problem that teachers can only manage with previous reflection on the validity of using approximated measurement in geometric constructions.

Such situations emerging from teaching practice on a specific concept are very important to connect the different instantations of it. The existence of this, and other similar situations lead us to define critical points, which require some interpretive analysis. A critical point is a manifestation of the difficulties that the teacher has to deal with the mathematical object (In This case the perpendicular bisector) due to its complexity. They are explicit in the form of errors, omissions, inaccuracies or lack of logical consistency in the teacher's speech. It is important to note that critical points relate to teachers' decisions and probably to their knowledge of the content, but are not necessarily a consequence of its lack. Seeking for those indicators in the transcription of the lessons, we have identified two critical points related to teachers' knowledge about the bisector. The first one has to do with measurement, as was illustrated above. The second one is related to the lack of consistency between the process of constructing the bisector and the definition in use. Both of them are reflected in Figures 2,3, and 4 with an exclamation mark.

The construction procedure (PR_2) used in Antonia's class considers the bisector as the locus of points equidistant from the two ends of a segment. However, she defines the bisector as the perpendicular through the midpoint of a given segment (D_1). This lack of coherence has important consequences for making sense of the bisector.

FINAL REMARKS

Having tools to visualize practice on a particular mathematical concept is important to decompose the intricate system of definitions, properties, processes, etc. (instantiations) that teachers use to approach this concept. Knowing not only which those instantiations are, but also how and when they arise in the course of a class, how they relate to each other, and the difficulties they involve is critical to understanding teachers' mathematical knowledge. Furthermore, it is assumed that mathematical knowledge for teaching is a complex system of instantiations distributed among all professionals.

We have applied a method based on the model proposed by the ontosemiotic approach to visualize three different mathematics classes on the bisector. The results highlight the use of different definitions, construction procedures and processes, as well as two critical points that inform about the difficulties for connecting the definitions, properties or procedures used. From our research derives the close relationship between the teaching of the perpendicular and the fundamental problem of using direct measurement for geometrical constructions, conjecturing or proving. To define the bisector as the perpendicular line through the midpoint of a given segment has been associated with an automation of its geometrical construction. On the other hand, defining the bisector as a locus has been associated with a broader communication process, and the emergence of modeling.

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REFERENCES

Blum, W. (2002) "ICMI STUY 14; Applications And Modelling In Mathematics Education-Discussion Document", *Education Studies in Mathematics*, 51, pp. 149-171.

Davis, B. and Renet, M. (2013). Profund understanding of emergent mathematics: broadening the construct of teacher's disciplinary knowledge. *Educational Studies in Mathematics*, 82(2), 245-265.

Fernández, C. & Yoshida, M. (2004). *Lesson study: a Japanese approach to improving mathematics teaching and learning*. Mahwah, NJ: Erlbaum.

Godino, J. D. Batanero, C. y Font, V. (2007). The ontosemiotic approach to research in mathematics education. ZDM. The International Journal on Mathematics Education, 39
(1-2), 127-135.

Godino, J., Contreras, A. & Font, V. (2006). Análisis de procesos de instrucción basado en el enfoque ontológico-semiótico de la cognición matemática. *Recherches en Didactiques des Mathematiques*, 26 *(1)*, 39-88.

Malaspina, U. & Font, V. (2010). The role of intuition in the solving of optimization problems. *Educational Studies in Mathematics*, 75*(1)*, 107-130.

Martinho, M. H., y Ponte, J. P. (2009). Communication in the classroom: Practice and reflection of a mathematics teacher. *Quaderni di Ricerca in Didattica (Matematica)* Supplemento n. 2 al n. 19.

Pochulu, M. & Font, V. (2011). Análisis del funcionamiento de una clase de matemáticas no significativa. *Revista Latinoamericana de Investigación en Matemática Educativa-RELIME*, 14 *(3)*, 361-394.

Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8*(3)*, 255-281.

Sullivan, P. & Wood, T. (2008). *The International Handbook of Mathematics Teacher Education: Vol. 1. Knowledge and beliefs in mathematics teaching and teaching development.* Rotterdam, The Netherlands: Sense publishers.

Tomás Ferreira, R. (2005). *Portuguese student teachers' evolving teaching modes: A modified teacher development experiment*. Doctoral dissertation. Illinois State University, USA.