THE OBJECT METAPHOR AND SYNECDOCHE IN MATHEMATICS CLASSROOM DISCOURSE

VICENÇ FONT, JUAN D. GODINO, NÚRIA PLANAS, JORGE I. ACEVEDO

The distinction between the form in which mathematical objects exist and their ostensive representations is a topic of interest in the field of mathematics education, as it is shown by the work of Sfard (1994, 2000), Duval (2008), and Font, Godino and D'Amore (2007). This interest is understandable if one considers that the same mathematical object may have different representations and that there are representations that are well formed syntactically but which do not represent a mathematical object. Moreover, a material mathematical sign can be considered to represent, depending on what is most appropriate, a particular, general or ideal object.

In mathematical discourse it is considered, whether explicitly or implicitly, that mathematical objects exist in a special way (non-ostensive, virtual, ideal, mental, abstract, general, *etc.,* depending on the theoretical perspective) that is different from the way in which physical objects exist, and which particularly differs from the material symbols that represent them. In line with that stated by various authors (Sfard, 2000; Lakoff & Núñez, 2000; Acevedo, 2008; Presmeg, 1997) we assume that speaking about the existence of mathematical objects, as objects that exist in a form that is different from that of their material symbols, is essentially a metaphorical question. Somehow one of the properties of objects, such as chairs, trees, stones, *etc.,* is translated into the world of mathematical objects.

Mathematical discourse moves flexibly between representations and the mathematical objects they represent. In mathematical practice it is not always considered necessary to distinguish explicitly the representation from the object represented, as this distinction is taken for granted. However, at times it is worth making a clear distinction between the two of them; for example, when a new representation is introduced, when speaking of representations that are well-formed syntactically but which do not represent a mathematical object, or when it is necessary to distinguish between the particular or general nature of the object represented. In the case of the professional practice of mathematicians, the use of the object metaphor (explained later), as well as that of the synecdoche, which treats a particular case as if it was a generality, does not seem to create any conflict.

A basic aim of mathematics education is that students learn to move flexibly between representations and the mathematical objects they represent. This is not easy for students to learn, and it also poses a challenge for teachers because they are not always aware of the complexity of this "language game" (Wittgenstein, 1953). This article presents classroom vignettes that illustrate how students and teachers make use of representations of mathematical objects and how they refer to them in terms of existence. This involves a metaphorical discourse that, under certain circumstances, poses problems for students' understanding, and which may hinder, among other things, the processes of idealization and generalization.

The aim of this paper is to illustrate how teachers and students speak in class about mathematical objects and their representations. Some conflictive uses of a particular object to refer to a general object (synecdoche) are also discussed.

The object metaphor in teachers' discourse

The conceptual metaphor, "mathematical entities as physical objects", is a grounding ontological metaphor (Lakoff & Núñez, 2000), which we call the "object metaphor". The object metaphor is always present in the teacher's discourse because here the mathematical entities are presented as "objects with properties" that can be physically represented (on the board, with manipulatives, with gestures, *etc.*). In Acevedo (2008), metaphorical expressions [1] of the object metaphor occur when the mathematics teachers refer to the graph of a function as an object with physical properties. When they talk about the application of mathematical operations in order to obtain the first derivative of a function, they use verbal expressions and gestures that suggest the possibility of manipulating mathematical objects as if they were things with a physical existence (Acevedo, 2008, p. 137):

> Teacher 1: The derivative of the numerator, no! You multiply by the denominator as it is, minus the numerator multiplied by the derivative of the denominator. Okay. Now you divide it by the denominator … square, that's it. (…) This is the first derivative. Now, what's next? *To operate, to manipulate* … What's left?

The use of the object metaphor facilitates the transition from the ostensive representation of the object to an ideal and non-ostensive object. Hence, the use of this type of metaphor leads to talk in terms of the "existence" of mathematical objects. This use may lead students to assume that mathematical objects exist within the mathematical discourse (internal existence) and, at times, students may

30(1) - March 2010:29(1) - March 2009 FLM 23/12/09 9:39 PM Page 16

suppose that they exist like chairs and trees do (external existence, physical or real). The work by Acevedo (2008, pp. 136–137) includes a classroom discussion on the domain of the logarithm function and another on the domain of the square root function, both of which occur while teaching the graphical representation of functions.

In the following extract the use of the word "exist" is considered within the language game of the mathematical discourse, in contrast to the comment of the former teacher, which might suggest to students that the derivative of the function has an external existence:

> Teacher 2: The domain goes from zero to infinite because logarithms of negative numbers *do not exist;* the logarithm of minus one *does not exist*. Shall we consider it with the zero included? (…)
>
> Not the negative … because the square root of a negative number *does not exist*. [2] We could also say the real numbers without the negatives, or even easier, all the positive numbers … we can write it like this, with an interval, from zero to infinite … now the zero is included.

If the teacher does not take care when using the verb "exist", the students in this class may not remain within a position of "internal existence". Instead, they may change the "language game" (Wittgenstein, 1953) and assume the "external existence" or reality of mathematical objects. In the following excerpt a third and different teacher explains the graphical representation of functions to students and explicitly mentions the idea of existence, though he does so in a rather controversial way (Acevedo, 2008, p. 137):

> Teacher 3: Then … *this function always exists,* the domain will be all real numbers and there won't be any vertical asymptotes.

Here there is a deviation from the "expected" use of the word "exists" within the language game of the mathematics discourse. One could reasonably state that the images of the values in the domain exist or are defined. When attributing existence to the whole function instead of talking about its images, the teacher is using the word "exists" in a way that can lead to the understanding of the function as a "real" object with properties. Moreover, by doing so the teacher can promote the movement from the mathematical internal existence of the object to a physical external existence.

Distinction between ostensive and non-ostensive objects

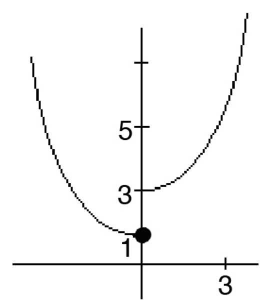
In this section we draw on the theoretical distinction between ostensive and non-ostensive objects as established by the onto-semiotic approach to mathematics education:

> Ostensive/non-ostensive mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated ostensive (notations, symbols, graphs, *etc.*). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation). (Godino, Batanero & Font, 2007, p. 131)

In mathematics discourse it is possible to talk about ostensive objects representing non-ostensive objects that do not exist. For example, we can say that $f'(a)$ does not exist because the graph of $f(x)$ has a pointed form at $x = a$. This gives us another example of the semiotic and discursive complexity of classroom discourse when referring to the existence of mathematical objects. In Acevedo (2008, p. 320) we find the following remark made by a teacher in the classroom:

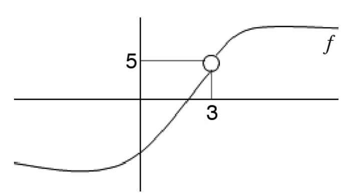
> Teacher 4: As you can see, the one-sided limits are not the same and then the limit does not exist… or the limit is infinite, I mean it is plus or minus infinite.

In García (2008, App. 2, p. 10) we find a teacher who refers to ostensive objects that represent non-ostensive objects that do not exist:



> Teacher 5: The function is defined in 0 … the one-sided limits exist but they are different. When the limit goes to zero from the left it is one. When the limit goes to zero from the right, it is 3. What is the limit when it goes to 0?
>
> Students: You can't know it. There is not … But, if it has two …
>
> Teacher 5: We saw that the one-sided limits exist for $f(x)$ when x goes to 3 and it is 5 [referring to previous work]. Now *the limit of f(x) when x goes to 0 does not exist.*

In the next transcript (García, 2008, App. 2, p. 8), the same teacher uses a discourse with ostensive objects, $f(3)$, that represent non-ostensive objects that do not exist; in this case, he does not say that they do not exist but literally says, "We cannot have them".



> Teacher 5: Let's imagine this function [the symbol   is used to indicate that the function is not defined at x = 3]. What is the domain of f? [He writes on the board.]

> And $f(3)$? Don't make the mistake of saying five, because it is not in the domain and *we cannot have an image*. We are not worried about $f(3)$, but about going as close as possible to three, before and after the three. Where are the images? Now I don't have a formula.
>
> Students: Near the five.
>
> Teacher 5: And now if I get closer to three on the right, where are the images?
>
> Students: Above five.
>
> Teacher 5: Yes, we can say limit of $f(x)$ when x goes to three.
>
> Students: But $f(3)$ does not exist.
>
> Student: But the asymptote does not touch it either.
>
> Teacher 5: It is curious but $\lim_{x\to 3} f(x) = 5$ [on the board]. It is not defined at three but its limit does exist. That limit exists without having the analytical expression and *without having* $f(3)$.

To talk about the existence of certain non-ostensive objects we have to use a discourse with ostensive objects that are constituted in accordance with the "grammar" that regulates the construction of well-established formulas. This type of discourse is frequently used by many students, as the following excerpt shows (Acevedo, 2008, p. 376):

> Student: (…) but since there are vertical asymptotes … they're right here … so *neither the derivative nor the function exists,* they don't exist …

The use of ostensive objects which represent non-ostensive objects that do not exist may create confusion in students' thinking, though it can also lead them to make philosophical reflections. This is the case with one student (Acevedo, 2008, p. 213) who distinguished "to be" and "to exist" when he misunderstood the vertical asymptote:

> Teacher 6: Could you explain a bit more about the vertical asymptote?
>
> Student: The vertical asymptote *is* the value *that does not exist* in the function.

The existence of well-established ostensive objects that represent non-ostensive objects that do not exist facilitates the consideration of the non-ostensive object as different from the ostensive object that represents it. Duval (1995, 2008) has pointed to the importance of different representations and transformations between representations in the students' understanding of the mathematical object as something different from its representation.

Many textbooks of mathematics make students observe that an object has many different representations and that it is necessary to distinguish the object from its representation. For instance, a Catalan textbook (Barceló *et al.,* 2002, p. 89) says:

> In all the activities carried out, you have been able to observe the different ways of expressing a function: as a statement, as a table of values, as a formula and as a graph. You must always remember these four forms of representation and know how to go from one to another.

However, textbooks often tend to identify the mathematical object with one of its representations. In the same textbook (Barceló *et al.,* 2002, p. 90), it is said, "Given the function $f(x) = 1/x$ …". Here the representation is identified with the object or differentiated from it depending on the purpose. Peirce (1901/1978, §2.273) mentions this idea:

> To stand for, that is, to be in such a relation to another that for certain purposes it is treated by some mind as if it were that other. Thus, a spokesman, deputy, attorney, agent, vicar, diagram, symptom, counter, description, concept, premise, testimony, all represent something else, in their several ways, to minds who consider them in that way.

In mathematical practices we constantly identify the object with its representations and, on the other hand, we make a distinction between the object itself and some of its representations. The rules of this language game, where the object metaphor is crucial, may be difficult to learn for some students. When we deal with physical objects we can differentiate the sign from the object (for instance, the word "watch" and the physical object "watch"). The object metaphor as used in mathematics discourse enables this differentiation to be transferred to mathematical objects and, therefore, we also differentiate the "representation" from the "mathematical object". Moreover, the type of discourse we produce in the mathematics classroom leads to infer the "existence" of the object as something independent from its representation. This situation allows us to draw conclusions about the existence of a mathematical object that can be represented by means of different "representations".

Synecdoche in the 'particular-general' relationship

Mathematical reasoning, going from the general to the general, introduces an intermediate phase that consists of contemplating a particular object. For example, in the demonstration of Pythagoras' Theorem one says, "Given a right-angled triangle ABC …", or in stating the definition of the derivative function one says, "Given a function $f(x)$ …". However, this fact poses a serious dilemma: if reasoning has to be applied to a specific object (for example, a triangle) it is necessary to have some guarantee that we reason about any object so that the generalization in which reasoning ends can be justified. Furthermore, since the specific object is linked to its representation there is also the problem of whether the representation refers to a specific object or a general concept, given that the ostensive representation is a particular material object (Font, Godino & D'Amore, 2007; Font, Godino & Contreras, 2008). An example of the difficulties faced by students when dealing with generic elements can be observed in the following dialogue, collected by Planas in one of her observations in a mathematics classroom:

> Teacher 7: What is the sum of the angles of a triangle? What is the sum of the angles of a quadrilateral? Or those of a pentagon? Or a hexagon? Or those of any polygon?

To clarify the statement of the problem, the teacher asks the students what they have understood by the expression "any polygon". The first student to answer says:

> Student: Any polygon means that I can choose the polygon I want, so I choose the triangle because it's the easiest… it always gives 180°. I could have chosen the octagon, the square or the pentagon, but I think everybody would have chosen the triangle… there's no point making life complicate.

The "onto-semiotic approach" to knowledge and mathematics teaching (Godino, Batanero & Font, 2007) considers two pairs of contextual attributes of mathematical objects which we believe are useful for describing and explaining the difficulties that students have in understanding how to use generic elements in classroom mathematics discourse. These attributes are the distinctions 'extensive/intensive' and 'expression/ content' (semiotic function).

> Extensive – intensive (example – type):
> an extensive object is used as a particular case (a specific example, *i.e.,* the function $y = 2x + 1$) of a more general class (*i.e.,* the family of functions $y = mx + n$), which is an intensive object.
>
> Expression – content:
> these are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are not isolated, but rather related to mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These two dualities become useful when analyzing the complexity associated with the use of the generic element. The use of the generic element is associated with a network of semiotic functions (and, therefore, representations) that relate intensive to extensive objects. These semiotic functions have in common that they all are of the representational type, in the sense that they facilitate the representation of the expression of content. They can also be of different types according to whether the expression or the content are extensive or intensive, and according to the correspondence criterion between the expression and the content. For example, when a subject relates an object to the class to which it belongs, this semiotic function is, on the one hand, representational, in that the particular case can be taken as representative of the class. However, it is of the metonymic type (specifically, a synecdoche – Presmeg, 1998), since an extensive (a part) is taken for the whole (the class); in this case the correspondence criterion is that of "belonging", while in other cases the semiotic functions are exclusively representational.

The analysis of dialogues between teachers and students has enabled us to detect some of the characteristics of the language game regarding generic elements and the difficulties which students may face in taking the rules of this game. Font and Contreras (2008, pp. 44–45) illustrate one student's difficulty in reasoning with generic elements and show how the teacher explains the rules that govern the use of the generic example. In the dialogue below, the teacher asks to solve the following activity from their textbook: "Exercise: Given the function $f(x) = ax + b$, show that $f'(x_0) = a$, independently of the value x_0". This activity was set just after the teacher has explained a written paragraph where the derivative of the function $f(x) = k$, $f'(x) = 0$, is justified first graphically, reasoning on the slope of the tangent line at any point of the straight line, and then calculating the limit of the average rate of change of the function f.

> Teacher 8: You are going to do it in two different ways: graphically and using limits, okay? And then someone will come out to the blackboard and correct it. Meanwhile I will be giving out some materials that will be useful afterwards, and so you will have them ready.
>
> Student: But graphically, we can… This is an example, if we represent it graphically it is an example.
>
> Teacher 8: (while nodding and confirming the student is right and approaching him) Yes, that's it!
>
> Student: And it says that x zero is considered.
>
> Teacher 8: Yes, but to prove it, take any point on this straight line, any one at all, and do it, and as you can do this with any point and with any straight line, it will help to prove it. Okay? But you are right, of course, to be able to draw you have to choose a specific point and a specific line.

It is essential that teachers properly "control", among other aspects, the use of metonymy (dialectic between particular and general objects) when they explain, whether explicitly or implicitly, the rules that govern the use of the generic element.

Final Remarks

Some authors, such as Duval (2008), highlight the need for students to distinguish between the mathematical object and its representations if they are to develop a good understanding of mathematics. They believe it is crucial for the student to manipulate different representations of the same object. In this paper we have illustrated other aspects of the process through which students come to consider the reality of mathematical objects that are different to their representations, as well as the use of these different representations. These other aspects to take into account are: (1) the role of the object metaphor; and (2) the discourse about ostensive objects that represent non-ostensive objects which do not exist. A further issue that has been considered is whether the represented object is a particular or general object, since the ostensive representation is a particular material object.

We have argued that the object metaphor plays a central role in the classroom discourse in which teachers and students talk about mathematical objects and physical entities. We have shown how the use of metaphorical expressions of object metaphors in the mathematics classroom discourse leads students to interpret conceptual mathematical entities as "objects that exist". On the other hand, the mathematics discourse about ostensive objects representing non-ostensive objects that do not exist, and about the identification (differentiation) of the mathematical object with one of its representations, leads students to regard mathematical objects as being different from their ostensive representations. As a consequence, the classroom discourse could help develop students' understanding of non-ostensive mathematical objects as objects that exist independently of their representations.

The use of the object metaphor and the distinction between ostensive and non-ostensive objects described in this paper are crucial aspects when aiming to answer the following question: By what process in the classroom do students become convinced that there are mathematical objects which exist independently of people and of the language we use to describe them?

Other aspects that also play a role in generating this realist [3] (platonic or empiricist) conception of mathematics include: 1) the discourse that considers mathematics as an objective science; 2) the discourse about the predictive success of sciences that make use of mathematics; and 3) the process of reification and the convenience and simplicity that follows from postulating the existence of mathematical objects. These aspects will be dealt with in a future article that will address the following question: Why and how are mathematical definitions and properties considered to be definitions and properties of mathematical objects that exist in a form which is different from that of their material representations?

Notes

[1] Conceptual metaphors enable metaphorical expressions to be grouped together. A metaphorical expression, on the other hand, is a particular case of a conceptual metaphor. For example, the conceptual metaphor "the graph is a path" appears in classroom discourse through expressions such as "the function *passes* through the coordinate origin" or "if *before* point M the function is ascending and *after* it is descending then we have a maximum". The teacher is unlikely to say to students that "the graph is a path" but, rather, will use metaphorical expressions that suggest this.

[2] At this school level, in our country, students are not taught about complex numbers.

[3] Realism states that there are mathematical objects that exist independently of people (in the Platonic world or in nature). In the Platonic view, the reality described by mathematical objects does not refer to the material objects that form part of our experiential world but, rather, to ideal, non-empirical objects (which cannot be perceived by the senses); these latter objects are perfect (determined with complete precision), immutable (totally permanent) and absolutely objective (totally independent of thought and perception). In the empiricist (holistic) view there are mathematical objects with properties and whose existence is independent of any knowing subject. The different notions of 'objects that exist' can be graded as follows: the only objects that exist independently of people are macroscopic material objects such as tables or trees, *etc.;* existence can be attributed to non-observed objects which were observable at another point in time, *e.g.* dinosaurs; there are entities whose existence enables us to observe certain phenomena, *e.g.* electrons; and the existence of non-observable entities such as non-ostensive mathematical objects.

Acknowledgment

The research work reported in this article was carried out as part of the following projects: SEJ2007-60110/EDUC, MEC-FEDER, and EDU 2009-08120/EDUC.

References

Acevedo, J. I. (2008) *Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones* [Phenomena related to the use of metaphors in the teacher's discourse. The case of the graphs of functions]. Unpublished doctoral dissertation, Barcelona, ES, Universitat de Barcelona.

Barceló, R., Bujosa, J. M, Cañadilla, J. L., Fargas, M. and Font, V. (2002) *Matemàtiques 1.* Barcelona, ES, Castellnou.

Duval, R. (1995) *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels,* Berne, CH, Peter Lang.

Duval, R. (2008) 'Eight problems for a semiotic approach in mathematics education', in Radford, L., Schubring, G., and Seeger, F. (eds.), *Semiotics in mathematics education: epistemology, historicity, classroom, and culture*. Rotterdam, NL, Sense, pp. 39–62.

Font, V. and Contreras, A. (2008) 'The problem of the particular and its relation to the general in mathematics education', *Educational Studies in Mathematics,* **69**(1), 33–52.

Font, V., Godino, J. D. and Contreras, A. (2008) 'From representations to onto-semiotic configurations in analysing the mathematics teaching and learning processes', in Radford, L., Schubring, G., and Seeger, F. (eds.), *Semiotics in mathematics education: epistemology, historicity, classroom, and culture.* Rotterdam, NL, Sense, pp. 157–173.

Font, V., Godino, J. D. and D'Amore, B. (2007) 'An onto-semiotic approach to representations in mathematics education', *For the Learning of Mathematics,* **27**(2), 2–7.

Font, V., Godino, J. D., Planas, N. and Acevedo, J. I. (in press) 'The existence of mathematical objects in the classroom discourse', *Proceedings of the Sixth Conference of the European Society for Research in Mathematics Education,* Lyon, FR, February 2009.

García, M. (2008) *Significados institucionales y personales del límite de una función en el proceso de instrucción de una clase de primero de bachillerato* [Institutional and personal meanings given to the limit of a function during the teaching process in a classroom with students aged 16 to 17]. Unpublished doctoral dissertation, Jaén, ES, Universidad de Jaén.

Godino, J. D., Batanero, C. and Font, V. (2007) 'The onto-semiotic approach to research in mathematics education', *ZDM-The International Journal on Mathematics Education,* **39**(1–2), 127–135.

Lakoff, G. and Núñez, R. (2000) *Where mathematics comes from: how the embodied mind brings mathematics into being,* New York, NY, Basic.

Peirce, C. S. (1901/1978) 'Speculative grammar', in *The collected papers of Charles Sanders Peirce,* Hartshorne, C., Weiss, P. and Burks, A. (eds.), Cambridge, MA, The Belknap Press of Harvard University Press, §2.273.

Presmeg, N. C. (1997) 'Reasoning with metaphors and metonymies in mathematics learning', in English, L. D. (ed.), *Mathematical reasoning: analogies, metaphors, and images.* Mahwah, NJ: Lawrence Erlbaum Associates, pp. 267–279.

Presmeg, N. C. (1998) 'Metaphoric and metonymic signification in mathematics', *Journal of Mathematical Behavior,* **17**(1), 25–32.

Sfard, A. (1994) 'Reification as the birth of metaphor', *For the Learning of Mathematics,* **14**(1), 44–54.

Sfard, A. (2000) 'Steering (dis)course between metaphors and rigor: using focal analysis to investigate an emergence of mathematical objects', *Journal for Research in Mathematics Education,* **31**(3), 296–327.

Wittgenstein, L. (1953) *Philosophische Untersuchungen / Philosophical investigations,* New York, NY, MacMillan.